Available online at www.sciencedirect.com
ScienceDirect
Physica A 371 (2006) 513–524
PHYSICA A
www.elsevier.com/locate/physa

Valuation of stochastic interest rate securities with time-dependent variance

Javier Villarroel*

Facultad de Ciencias, Universidad de Salamanca, Plaza Merced s/n, Salamanca 37008, Spain

Received 11 January 2006; received in revised form 7 April 2006
Available online 19 May 2006

Abstract

We consider the problem of how to prize general securities whose payoff at maturity only depends on the interest rate r_T at the time of exercise, where r_t is supposed to be a stochastic Feller process. We show how to generalize the results of Cox et al. [Econometrica 53 (2) (1985) 385] regarding bond valuation to a situation where the stochastic evolution of r_t under the martingale probability involves time-dependent coefficients and the payoff is arbitrary. The solution to this problem is given in terms of the propagator for the heat operator with a potential. This propagator is constructed in terms of a classical harmonic oscillator with time-dependent frequency.

Keywords: Option and derivative pricing; Econophysics; Stochastic differential equations

1. Introduction

In this paper we consider the problem of how to prize generic European interest rate securities. These are financial instruments whose payoff at maturity time T depends *only* on the terminal value of the short interest rate r_T, i.e., the payoff function is $\Theta(r_T)$ where Θ is an *arbitrary* continuous function and T is the exercise time of the security. We assume that the evolution of r_t is described by the stochastic differential equation (SDE)

$$dr_t = a(t, r_t)\,dt + b(t, r_t)\,dW_t, \tag{1}$$

where W_t is a Brownian motion (BM) process and a and b^2 are assumed to be *affine* functions in x; namely they can be written as

$$a(t, x) = q(t) - 2m(t)x \quad \text{and} \quad b^2(t, x) = \sigma^2(t)x \tag{2}$$

for certain functions of time $q(t)$, $m(t)$ and $\sigma(t)$.

*Corresponding author.
E-mail address: javier@gugu.usal.es.

0378-4371/$ - see front matter © 2006 Elsevier B.V. All rights reserved.
doi:10.1016/j.physa.2006.04.070

The (actual) t-price $\mathrm{V}_{t,T}$ of the above security given that $r_t \equiv x$ is a known value is the expected value of the payoff function times the relevant discount factor:

$$\mathrm{V}_{t,T} = \mathbb{E}(\Theta(r_T)\mathrm{e}^{-\int_t^T r_s\,\mathrm{d}s}|r_t = x). \tag{3}$$

We suppose that the relevant expectation already discounts the market price of risk, i.e., that the expectation is taken with respect to the risk-neutral or martingale probability. Further based on general considerations on the conditional expectation, for given values of $t \leqslant T$, the price $\mathrm{V}_{t,T}$ can be written as a simple function of x, i.e., there exists a function $v_{t,T} : \mathbb{R} \to \mathbb{R}$ such that

$$\mathbb{E}(\Theta(r_T)\mathrm{e}^{-\int_t^T r_s\,\mathrm{d}s}|r_t = x) = v_{t,T}(x). \tag{4}$$

The function defined by the path integral (4) can be found as the solution of the final value problem

$$\left(\frac{b^2}{2}\frac{\partial^2}{\partial x^2} + a(t,x)\frac{\partial}{\partial x} - x + \frac{\partial}{\partial t}\right)v_{t,T}(x) = 0; \quad t < T, \ \ v_{T,T}(x) = \Theta(x). \tag{5}$$

This is the content of the Feynman–Kac formula, a generalization of the well known connection between the Schrödinger operator and the Feynman path integral of quantum mechanics (see, for example, Ref. [1]).

Setting $\Theta = 1$ yields the price at t of a bond with maturity T, denoted as $p_{t,T}(x)$. If, in addition, we assume constant coefficients this is the classical Cox–Ingersoll–Ross (CIR) model of Ref. [2].

To clarify why interest rate derivatives have such an importance we note the following. Unlike the interest rate itself, which is not tradeable on any established securities market, different bonds can be traded on a recognized investment exchange, as bond owners may decide to sell them at any time before maturity. Thus, in this setting the basic *tradeable* asset is the bond process $p_{t,T}(r_t)$. Consider at time t a security, say an option, written on the value at T of a (zero-coupon) bond maturing at $\tau, t < T < \tau$, with payoff function $\tilde{\Theta}(v_{T,\tau})$ where $v_{T,\tau}$ is the T-price of the bond; using (4) we have that $v_{T,\tau} = p_{T,\tau}(r_T)$ for some $p_{T,\tau} : \mathbb{R} \to \mathbb{R}$ and hence the payoff also equals

$$\tilde{\Theta}(v_{T,\tau}) = \tilde{\Theta}(p_{T,\tau}(r_T)) \equiv \Theta_{T,\tau}(r_T).$$

Thus, *any security on a bond maturing at τ with payoff $\tilde{\Theta}$ can be viewed as a security on r_T* with associated payoff $\Theta_{T,\tau}(r_T) \equiv \tilde{\Theta}(p_{T,\tau}(r_T))$ where $p_{t,\tau}(x)$ solves (5) and $p_{\tau,\tau}(x) = 1$.

Classical examples relevant in Finance and associated payoffs are:

(i) The zero-coupon bond itself if sold by the owner prior to maturity: if the bond matures at τ and is sold at time $T \leqslant \tau$ then the payoff function is given by $\tilde{\Theta}(y) = y$ and $\Theta(x) = p_{T,\tau}(x)$.

(ii) *Call European option* to buy a bond at T with strike k and maturity $\tau > T$. Here

$$\tilde{\Theta}(y) = (y - k)^+, \quad \Theta(x) = (p_{T,\tau}(x) - k)^+,$$

where we recall that the positive part of a function f is defined as

$$f^+(x) = f(x) \ \text{if} \ f(x) \geqslant 0, \quad f^+(x) = 0 \ \text{otherwise}.$$

(iii) *Put European option*: $\tilde{\Theta}(y) = (k - y)^+, \ \Theta(x) = (k - p_{T,\tau}(x))^+$.

(iv) *Caps*. In terms of a certain constant L they are defined by

$$\tilde{\Theta}(y) = L\left(\frac{1}{y} - 1\right)^+, \quad \Theta(x) = L\left(\frac{1}{p_{T,\tau}(x)} - 1\right)^+.$$

(v) Options on coupon-paying bonds. Suppose the bond has n payments at times $t_1 < \cdots < t_n$ and maturity $T \geqslant t_n$; then it is well known that $\Theta(x)$ is a linear combination of payoff functions corresponding to case (ii).

These examples underscore the interest of having closed formulas to value general interest rate securities, as our results do; this is in contrast with the original CIR paper where *only* valuation of bonds is considered.

Unfortunately, there are very few choices of the functions a and b for which a solution of (5) can be obtained in closed form. An exception is given by the classical Vasicek model [3], wherein r_t is assumed to be an Ornstein–Uhlenbeck Gaussian process. A serious drawback of the model is that the interest rate can take negative values with positive probability. (For option pricing of American options in the framework of Vasicek model see Refs. [4–6].) To overcome this difficulty CIR consider an equilibrium economy in which $p_{t,T}(r_t)$ solves (5) and r_t evolves via Feller's SDE (1), (2) where σ, q, m and Θ are constants [2]. The resulting SDE (1) is named after R. Feller who first studied the transition density and nature of boundaries of a such process [7] and proved that solutions are positive. (It is interesting to point out that this SDE has also been recently used to model thermal reversal of magnetic fields in a magnetic grain [8].)

Due to overall simplicity and appealing properties the CIR model has become, to both academics and practitioners, the prototype one-factor model to describe interest rate dynamics and bond prices. A closed formula for bond prices was derived by CIR by solving (5) for $\Theta = 1$ (see Eq. (45)). This formula is currently used by major institutions like the Canada Department of Finance and the Danish National Bank to value bond prices (see Refs. [9,10]). However, unlike to what happens for the Vasicek model, r_t is no longer a Gaussian process which entails important complications; in particular, the problem of valuing general securities within this framework is, to our knowledge, *unsolved*.

A further difficulty, present in both the Vasicek and CIR models, is the impossibility to fit *observed bond prices* of all maturities, say, with those predicted from theory (i.e., from Eq. (45)). Motivated by this, Hull and White [11] advocate using a time-dependent version of the former models as an approach that allows *fitting the model to the observed term structure* (i.e., to real market data) (see also Refs. [12,13]). Such an assumption is reasonable from a physical perspective as one expects the local volatility σ to change in time. Hull and White give an explicit expression for the bond price within the generalized Vasicek model but, regretfully, they fail to solve the relevant PDE (5) in the CIR case, not even for the simplest case of bond pricing. Actually, such a solution has been so far elusive and it is common belief that the time-dependent CIR/Hull and White model is not amenable to analytic treatment (see, for example, the discussion in Refs. [14,15]).

In this paper we show that, in spite of the above claims, *analytic valuation of general European securities in the time-dependent CIR model is possible* whenever $q(t) = \sigma^2(t)/4$. We find a transformation that permits solving the SDE (1) and, in addition, converts (5) into a heat equation in the field of a time-dependent frequency harmonic oscillator. Alternatively, into a non-stationary Schrödinger equation in imaginary time. (Note that this "Wick rotation" of time is understood whenever we speak of the Schrödinger operator.) Concretely, assuming that both the coefficients m, σ and the boundary condition Θ are *arbitrary* functions, in Section 2 we (i) solve explicitly the SDE (1) and (ii) *give the solution to* (5) *in terms of the Green's function for the time-dependent Schrödinger operator of a harmonic oscillator with time-dependent frequency*. We next express this propagator in terms of the solution of the *classical* harmonic oscillator. Thus our results express the solution to (5) in terms of the solution of a *linear ordinary differential equation* (ODE) (see Eq. (25)).

Explicit constructions corresponding to concrete time dependency are considered in Section 3. In particular, with constant coefficients, we *present the solution to the problem of valuation of general securities under the CIR model in an explicit way* (Eq. (43)).

For complete accounts of option pricing and stochastic calculus from the physicist and, respectively, economist, points of view see Refs. [16–20] and [1,12,21–24]. Complete information on historical series of USA treasury bonds and of yield charts is available on-line, see Refs. [25,26]. There exist countless commercial software tools of interest to investors based upon the CIR and Hull and White models; see, for example, Ref. [27].

We note that in the context of real markets new features may appear like the existence of self-scaling and long memory effects. To account for those effects several authors discuss the possibility that the dynamics of interest rates and of risky assets involve Levy process or fractional BM and discuss option pricing in such a framework (see Refs. [16,28–32]).

2. Claim's valuation and the non-stationary Schrödinger/heat operator

2.1. Interest rate evolution

As we have already pointed out we aim to solve both the SDE (1) and the PDE (5) when the coefficients satisfy (2) with $q(t) = \sigma^2(t)/4$. It turns out that there exists a coordinate transformation which allows solving both equations. We first consider Eq. (1) assuming that σ and m are arbitrary continuous functions, i.e., we determine the solution r_t to the nonlinear Itô's SDE

$$\mathrm{d}r_t = (\sigma^2(t)/4 - 2m(t)r_t)\,\mathrm{d}t + \sigma(t)\sqrt{r_t}\,\mathrm{d}W_t. \tag{6}$$

Let $M(t) \equiv \int^t m(z)\,\mathrm{d}z$ and let Z_t be defined by the Itô's integral

$$Z_t \equiv \int^t \sigma(z)\mathrm{e}^{M(z)}\,\mathrm{d}W_z. \tag{7}$$

We shall now prove that the solution to (6) is given by

$$r_t = \frac{\mathrm{e}^{-2M(t)}}{4}(Z_t + c)^2, \tag{8}$$

where c is an arbitrary constant.

To this end we define $\tilde{r}_t \equiv g(t, Z_t)$ where

$$g(t,z) = \frac{\mathrm{e}^{-2M(t)}}{4}(z + c)^2. \tag{9}$$

Then, noting that $\mathrm{d}Z_t = \sigma(t)\mathrm{e}^{M(t)}\,\mathrm{d}W_t$ it follows by application of Itô's rule (note that $g(t,z)$ is of class $C^{1,\infty}$) that $\tilde{r}_t$ solves

$$\begin{aligned}\mathrm{d}\tilde{r}_t &= \left(\partial_t + \frac{1}{2}\sigma^2(t)\mathrm{e}^{2M(t)}\partial_{ZZ}\right)g(t,Z)\,\mathrm{d}t + \sigma(t)\mathrm{e}^{M(t)}\partial_Z g(t,Z)\,\mathrm{d}W_t \\ &= \left(-2m(t)\tilde{r} + \frac{\sigma^2(t)}{4}\right)\mathrm{d}t + \frac{\sigma(t)}{2}\mathrm{e}^{-M(t)}(Z_t + c)\,\mathrm{d}W_t = \left(-2m\tilde{r} + \frac{\sigma^2}{4}\right)\mathrm{d}t + \sigma\sqrt{\tilde{r}_t}\,\mathrm{d}W_t\end{aligned}$$

which proves that the solution to (6) is given by (8).

Remarks. 1. With an analysis similar to that of Ref. [33] it can be proved that the condition $q(t) = \sigma^2(t)/4$ is necessary for (6) to have a solution with the representation $r_t = g(t, \int_{t_0}^t f(s)\,\mathrm{d}W_s)$ for some $g : [t_0, \infty) \times \mathbb{R} \to \mathbb{R}$ and $f : [t_0, \infty) \to \mathbb{R}$.

2. Motivated partially by Remarks 3 and 4, in the sequel we define a coordinate transformation $(x,t) \to (x^*, t^*)$ as follows:

$$t^* = \varphi_1(t) \equiv \int^t (\mathrm{e}^M\sigma)^2(t')\,\mathrm{d}t', \quad x^* = \varphi_2(t,x) \equiv 2\sqrt{x}\mathrm{e}^{M(t)}. \tag{10}$$

3. Note that $g(t,z) = \varphi_2^{-1}(t, z + c)$. Here and elsewhere a symbol $\varphi_2^{-1}(t, x^*)$ is used to mean the inverse function of φ_2 with respect to the spatial variable. Hence $\varphi_2^{-1}(t, \varphi_2(t,x)) = x$.

4. Further insight in solution (8) can be gained by noting that, by the well known Levy's characterization theorem (see Ref. [1]) and since φ_1 is strictly increasing, Z_t is also a BM ran in a different time, i.e., there exists a BM B_t such that $Z_t = B_{\varphi_1(t)}$. Hence $r_t = \varphi_2^{-1}(t, B_{\varphi_1(t)} + c)$.

5. In the constant-coefficients case Feller gives a solution in law of the SDE (6); this means that the transition density of the solution r_t is given but *not an expression of r_t in terms of W_t as* (8) *does*. Note that the transition density of r_t follows immediately from Remark 4.

2.2. Valuation of general interest rate securities

Suppose at a given time t we have $r_t = x$. Recall that the t-price $v_{t,T}(r_t)$ of a security that at maturity T pays $\Theta(r_T)$ solves (5); concretely

$$\left(\frac{\sigma^2(t)x}{2}\frac{\partial^2}{\partial x^2} + \left(\frac{\sigma^2(t)}{4} - 2m(t)x\right)\frac{\partial}{\partial x} - x + \frac{\partial}{\partial t}\right)v_{t,T}(x) = 0; \quad t < T, \quad v_{T,T}(x) = \Theta(x). \tag{11}$$

We next show how to solve this equation with general functions $\sigma^2(t),\ m(t),\ \Theta(x)$. Motivated by the method of solution of (6) we find it convenient to transform to coordinates (10); this map induces a corresponding transformation of functions $f \to f^*$ via

$$f^*(t^*, x^*) = f(t, x) \quad \text{or} \quad f^*(\varphi_1(t), \varphi_2(t, x)) = f(t, x). \tag{12}$$

By differentiation and use of the chain rule it follows that derivatives of $v_{t,T}(x) = v^*_{t^*,T^*}(x^*)$ are transformed as

$$\frac{\partial v_{t,T}(x)}{\partial t} = \sigma^2(t)\mathrm{e}^{2M(t)} \frac{\partial v^*_{t^*,T^*}(x^*)}{\partial t^*}$$

and so forth. Then, an easy calculation shows that $v^*_{t^*,T^*}(x^*)$ solves the final value problem

$$\left(\frac{1}{2}\frac{\partial^2}{\partial x^{*2}} + \frac{\partial}{\partial t^*} - V(t^*, x^*)\right) v^*_{t^*,T^*}(x^*) = 0, \tag{13.1}$$

with "final" condition

$$v^*_{T^*,T^*}(x^*) = \Theta(\varphi_2^{-1}(T, x^*)). \tag{13.2}$$

Here

$$V^*(t^*, x^*) \equiv \frac{\omega^{*2}(t^*)x^{*2}}{2}, \quad \omega^* = \frac{\mathrm{e}^{-2M^*}}{\sqrt{2}\sigma^*}. \tag{14}$$

Thus we have reduced the solution of (11) into that of a *heat equation in the field of a harmonic oscillator* with time-dependent frequency $\omega^*(t^*)$.

Let $G^*(T^*, X^*|t^*, x^*)$ be the Green's function or propagator for the heat (alternatively, non-stationary Schrödinger) operator in the potential field of a time-dependent harmonic oscillator (the reader may consult Ref. [34] for the basic facts of Green's function theory from a physicist point of view). The propagator solves the equation, in initial coordinates, $LG^*(T^*, X^*|t^*, x^*) = 0$ where

$$LG^* \equiv \left(\frac{1}{2}\frac{\partial^2}{\partial x^{*2}} + \frac{\partial}{\partial t^*} - V(t^*, x^*)\right) G^*(T^*, X^*|t^*, x^*) = 0, \tag{15.1}$$

$$G^*(T^*, X^*|T^*, x^*) = \delta(X^* - x^*) \tag{15.2}$$

and $\delta(X - x)$ is the Dirac delta function. Then, classical PDE theory (see Ref. [34]) yields that the solution to (11) is given by

$$v^*_{T^*,t^*}(x^*) = \int G^*(T^*, X^*|t^*, x^*)\Theta(\varphi_2^{-1}(T, X^*))\,\mathrm{d}X^* \equiv \int G^*(T^*, X^*|t^*, x^*)\Theta\left(\frac{\mathrm{e}^{-2M(T)}X^{*2}}{4}\right)\mathrm{d}X^*. \tag{16}$$

Recall that we use t, x as the *actual* time and coordinate while T, X are the final (maturity) time and forward coordinate. Hence $T^* = \varphi_1(T),\ X^* \equiv \varphi_2(T, X) = 2\sqrt{X}\mathrm{e}^{M(T)}$.

We next show how to solve (15) and construct such a propagator. Working with forward coordinates turns out to be more convenient. In this regard it can be proven that $G^*(T^*, X^*|t^*, x^*)$ *solves*, *in addition to* (15.1), *the adjoint equation*, in the forward variables, $L^\dagger G^* = 0$, or

$$\left(\frac{1}{2}\frac{\partial^2}{\partial X^{*2}} - \frac{\partial}{\partial T^*} - V^*(T^*, X^*)\right) G^*(T^*, X^*|t^*, x^*) = 0. \tag{17}$$

We skip the proof so as not to overload the article with mathematical details.

It is remarkable that this equation can be solved in closed form. To this end let $\lambda_1^*(T^*) \equiv \lambda_1^*(T^*|t^*)$ be the solution of the equation for a *classical* harmonic oscillator with variable frequency ω^*:

$$\frac{\mathrm{d}^2\lambda_1^*}{\mathrm{d}T^{*2}} - \omega^{*2}(T^*)\lambda_1^*(T^*) = 0 \tag{18.1}$$

satisfying, at $T^* = t^*$, the initial conditions

$$\lambda_1^*|_{T^*=t^*} = 1; \quad \dot{\lambda}_1^*|_{T^*=t^*} = 0. \tag{18.2}$$

(Here, $\dot{\lambda}^* \equiv \mathrm{d}\lambda^*/\mathrm{d}T^*$. Notice also the odd sign of (18.1), i.e., the "Wick-rotation into imaginary time".)

We next define $v(T^*, X^*|t^*, x^*)$ via

$$G^*(T^*, X^*|t^*, x^*) = \sqrt{\lambda_1^*}\,\exp\left(-\frac{\dot{\lambda}_1^*}{2\lambda_1^*}X^{*2}\right)v(T^*, X^*|t^*, x^*), \tag{19}$$

whereupon we find that $v(T^*, X^*|t^*, x^*)$ solves the Cauchy problem

$$\frac{\partial v}{\partial T^*} - \frac{1}{2}\frac{\partial^2 v}{\partial X^{*2}} + \frac{\dot{\lambda}_1^*}{\lambda_1^*}\frac{\partial(X^* v)}{\partial X^*} = 0 \tag{20}$$

with the initial condition $v(t^*, X^*|t^*, x^*) = \delta(X^* - x^*)$. Dots stand for time T^* derivatives.

We note the remarkable fact that, upon use of transformation (19), Eq. (17) is converted into an equation where the potential term *is no longer present*; hence the latter equation is amenable to be solved via Fourier transformation. To this end we introduce the Fourier transform $\phi(T^*, s|t^*, x^*)$ as

$$\phi(T^*, s|t^*, x^*) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}sX^*} v(T^*, X^*|t^*, x^*)\,\mathrm{d}X^*, \tag{21}$$

whereupon we find that $\phi(T^*, s)$ solves the *first order quasilinear* PDE

$$\frac{\partial \phi}{\partial T^*} + \frac{s^2}{2}\phi - \frac{\dot{\lambda}_1^*}{\lambda_1^*}s\frac{\partial \phi}{\partial s} = 0; \quad \phi(t^*, s|t^*, x^*) = \mathrm{e}^{\mathrm{i}x^* s}. \tag{22}$$

A solution can be found through the well known method of Characteristics as

$$\phi(T^*, s|t^*, x^*) = \exp\left\{\mathrm{i}x^* s\lambda_1^*(T^*) - \frac{s^2}{2}\lambda_1^*(T^*)\lambda_2^*(T^*)\right\},$$

where

$$\lambda_2^*(T^*) \equiv \lambda_1^*(T^*)\int_{t^*}^{T^*}\frac{\mathrm{d}l}{\lambda_1^{*2}(l)}. \tag{23}$$

It is easy to prove that $\lambda_2^*(T^*)$ satisfies Eq. (18.1) with initial conditions $\dot{\lambda}_2^*|_{T^*=t^*} = 1$; $\lambda_2^*|_{T^*=t^*} = 0$ and hence that it is a second independent solution to that equation. Further one proves easily the useful relationship $-\lambda_2^*\dot{\lambda}_1^* + \lambda_1^*\dot{\lambda}_2^* = 1$.

Upon inversion of the Fourier transform (21) and use of the latter relationship to simplify some terms, we finally obtain that the propagator for the heat operator with a harmonic time-dependent oscillator potential (14) is given by

$$G^*(T^*, X^*|t^*, x^*) = \frac{1}{\sqrt{2\pi\lambda_2^*}}\exp\left[-\frac{\dot{\lambda}_2^*}{2\lambda_2^*}\left(X^* - \frac{x^*}{\dot{\lambda}_2^*}\right)^2 - \frac{\dot{\lambda}_1^*}{2\dot{\lambda}_2^*}x^{*2}\right]. \tag{24}$$

Using (16) we obtain that, if $r_t = x$, the t-price $v_{t,T}(x)$ of a security that pays $\Theta(r_T)$ at maturity is given by

$$v_{t,T}(x) = \frac{1}{\sqrt{2\pi\lambda_2^*}}\int \Theta\left(\frac{\mathrm{e}^{-2M(T)}X^{*2}}{4}\right)\exp\left[-\frac{\dot{\lambda}_2^*}{2\lambda_2^*}\left(X^* - \frac{x^*}{\dot{\lambda}_2^*}\right)^2 - \frac{\dot{\lambda}_1^*}{2\dot{\lambda}_2^*}x^{*2}\right]\mathrm{d}X^*. \tag{25}$$

The reader may wish to consult Ref. [35] for related application of some of these ideas. A general construction of the propagator for the heat equation with an asymptotically constant potential in terms of eigenfunctions of the stationary Schrödinger operator is given in Refs. [36,37].

2.3. Valuation of bonds

The above construction of the Feynman–Kac propagator involves, as a starting point, the evaluation of the "transformed" functions and coordinates $t^*, x^*, \omega^*, \lambda_j^*$ which can be a long, and sometimes cumbersome, process. In the case of bond valuation ($\Theta(x) = 1$) there exists a direct, neat answer in terms of the physical coordinates t, x. Indeed, for $j = 1, 2$ we define $\lambda_j(T) \equiv \lambda_j(T|t)$ as above:

$$\lambda_j(T|t) = \lambda_j^*(T^*|t^*), \quad j = 1, 2. \tag{26}$$

Bearing in mind that

$$\frac{\partial \lambda(T)}{\partial T} = \sigma^2(T) e^{2M(T)} \frac{\partial \lambda^*(T^*)}{\partial T^*} \tag{27}$$

we find, upon substitution in Eq. (18.1), that the functions $\lambda_j(T)$ can be evaluated directly by solving the linear ODE

$$\frac{d^2\lambda}{dT^2} - 2Q(T)\frac{d\lambda}{dT} = \frac{\sigma^2(T)}{2}\lambda(T) \tag{28.1}$$

with initial conditions

$$\dot{\lambda}_2|_{T=t} = \sigma^2(t) e^{2M(t)}; \quad \lambda_1 - 1 = \dot{\lambda}_1 = \lambda_2|_{T=t} = 0. \tag{28.2}$$

Here

$$\dot{\lambda} \equiv \frac{d\lambda}{dT} \quad \text{and} \quad Q(T) \equiv \partial_T \log \sigma(T) + m(T). \tag{29}$$

Finally, if $\Theta(x) = 1$, integral (25) can be evaluated in an exact way by completing squares in the exponent as

$$p_{t,T}(x) = \int G^*(T^*, X^*|t^*, x^*)\, dX^* = \frac{1}{\sqrt{\dot{\lambda}_2^*}} \exp\left[-\frac{\dot{\lambda}_1^*}{2\dot{\lambda}_2^*} x^{*2}\right] = \frac{\sigma(T) e^{M(T)}}{\sqrt{\dot{\lambda}_2(T)}} \exp\left[-2e^{2M(t)} \frac{\dot{\lambda}_1(T)}{\dot{\lambda}_2(T)} x\right]. \tag{30}$$

Remarks. 1. As we have already pointed out, the problem of pricing bonds under the time-dependent CIR model has been reduced in Ref. [11] to solving a system of *nonlinear* differential equations while the more general problem of valuation of arbitrary securities *remained unsolved. In contrast*, (25) *and* (30) *solve both problems in an explicit way*.

2. It is possible to prove that the functions λ_j, $j = 1, 2$ and their derivatives are positive and strictly increasing; furthermore they satisfy the identities

$$\frac{\dot{\lambda}_1(T)}{\dot{\lambda}_2(T)} = \int_t^T \frac{\sigma^2 f^2(s)}{\dot{\lambda}_2^2}\, ds, \quad \lambda_1\dot{\lambda}_2 - \dot{\lambda}_1\lambda_2 = \sigma^2(t) e^{2M(t)}.$$

This implies that p satisfies $0 \leqslant p_{t,T}(x) \leqslant 1$. Besides, the bond price decreases with the maturity time T from 1 to a limit value $\geqslant 0$; further p is a convex, decreasing to 0, function of x.

3. We note that the market convention is to quote bond-prices in terms of the *yield to maturity* $Y_{t,T}(r_t) \equiv -(1/(T-t)) \log p_{t,T}(r_t)$. Here we have

$$Y_{t,T}(r_t) = \frac{2}{T-t} e^{2M(t)} \frac{\dot{\lambda}_1(T)}{\dot{\lambda}_2(T)} r_t - \frac{1}{T-t} \log\left(\frac{\sigma(T) e^{M(T)}}{\sqrt{\dot{\lambda}_2(T)}}\right). \tag{31}$$

We recall that the yield curve $T \to Y_{t,T}(r_t)$ represents $Y_{t,T}(r_t)$ in terms of maturity T for given values of t and $r_t = x$. A normal (inverted) curve is one in which longer maturity bonds have a higher (smaller) yield compared to shorter-term bonds; thus such a yield curve slopes upwards (downwards).

3. Some reductions and concrete constructions

Here we shall consider some further reductions obtained by assuming that the functions $m(t)$, $\sigma(t)$ satisfy certain restrictions. Under these reductions the solutions to (18) and all objects appearing in the construction above can be constructed in a fully explicit way.

3.1. Constant frequency

We consider first the case where the functions $a(t,x)$, $b(t,x)$ of (2) can be written as $b(t,x)=\sigma(t)\sqrt{x}$, $a(t,x)=\sigma^2(t)/4+(\partial_t\sigma/\sigma)x$ for some $\sigma(t)>0$. This corresponds to m and σ being related by $m(t)=\partial_t\sigma/2\sigma$. Then, with an appropriate choice of integration constants we have that $M(t)=-\frac{1}{2}\log(\sqrt{2}\sigma(t))$; further, from (10) we find that

$$t^*\equiv\varphi_1(t)=\frac{1}{\sqrt{2}}\int_0^t\sigma(s)\,\mathrm{d}s,\quad x^*=\varphi_2(t,x)=2^{3/4}\sqrt{\frac{x}{\sigma(t)}}. \tag{32}$$

It is interesting to remark that, using (8) and ensuing remarks, the interest rate can be written in terms of a BM B_t as $r_t=R_t^2/4$ where $R_t\equiv\mathrm{e}^{-M(t)}(B_{\varphi_1(t)}+c)$ is normally distributed: $R_t\sim\mathcal{N}(\mu_t,v_t)$ with mean and variance

$$\mu_t=c\mathrm{e}^{-M(t)}=2^{1/4}\sigma^{1/2}(t)c,\quad v_t=\mathrm{e}^{-2M(t)}\varphi_1(t)=\sigma(t)\int_0^t\sigma(s)\,\mathrm{d}s.$$

Thus, the interest rate has mean value

$$\bar{r}_t=\left(c^2+\int_0^t\frac{\sigma(s)\,\mathrm{d}s}{\sqrt{2}}\,\mathrm{d}s\right)\sigma(t)\frac{\sqrt{2}}{4}. \tag{33}$$

Further, (32) implies that in this case the "frequency" ω^* of (9) is simply $\omega^*=1$ and hence Eq. (18) is a constant-coefficients linear ODE. We trivially obtain the solutions $\lambda_{1,2}^*$ of (18.1), (18.2) as

$$\lambda_1^*(t^*)=\cosh(T^*-t^*),\quad \lambda_2^*(t^*)=\sinh(T^*-t^*). \tag{34}$$

The value of any security follows from (25), (32) and (34). In particular, we get from (30) the price of the bond as

$$p_{t,T}(x)=\left(\cosh\int_t^T\frac{\sigma(s)\,\mathrm{d}s}{\sqrt{2}}\right)^{-1/2}\exp\left[-\frac{\sqrt{2}x}{\sigma(t)}\tanh\int_t^T\frac{\sigma(s)\,\mathrm{d}s}{\sqrt{2}}\right]. \tag{35}$$

The behavior of $p_{t,T}$ for long maturity times depends on whether $\sigma(t)$ is or not integrable. In terms of $\sqrt{2}B\equiv\int_t^\infty\sigma(s)\,\mathrm{d}s$ we have

$$\lim_{T\to\infty}p_{t,T}=\frac{1}{\sqrt{\cosh B}}\exp\left[-\frac{\sqrt{2}x}{\sigma(t)}\tanh B\right]=\begin{cases}>0 & \text{if } B<\infty,\\ 0 & \text{if } B=\infty.\end{cases}$$

Yields are easily evaluated. In particular, the yield to perpetuity $\lim_{T\to\infty}Y_{t,T}(r_t)$ is given by

$$\lim_{T\to\infty}Y_{t,T}(r_t)=\lim_{T\to\infty}\frac{\sigma(T)}{2\sqrt{2}}. \tag{36}$$

Thus, if the volatility satisfies $0<\lim_{T\to\infty}\sigma(T)\equiv\sigma_\infty<\infty$ so it does the long-term yield $Y_{t,\infty}$. Actually, condition $0<\sigma_\infty$ must hold for the yield curve to be normal. If $\sigma_\infty=0$ the yield curve must be inverted. A partial interpretation for this behavior follows from (33): if $0<\sigma_\infty$ the interest rate drifts on the real positive axis and fails to converge for long times. If $\sigma_\infty=0$ such that $\sigma(t)$ converges to zero faster than $O(t^{-1/2})$ the interest rate converges to zero and hence long-term investments give smaller yields.

In Fig. 1 we plot both $p_{t,T}$ and $Y_{t,T}$ in terms of maturity T for $t=1$ and $x=1/(2\sqrt{2})$ corresponding to the case $\sigma(t)=3\sqrt{2}+\sqrt{2}/(1+t^2)$. Notice how the bond price decreases to zero while the yield curve is normal (slopes upwards towards the perpetuity yield: $Y_{1,\infty}(1/(2\sqrt{2}))=3/2$) in agreement with previous comments: here condition $0<\sigma_\infty\equiv3\sqrt{2}<\infty$ holds and, in particular, σ is not integrable.

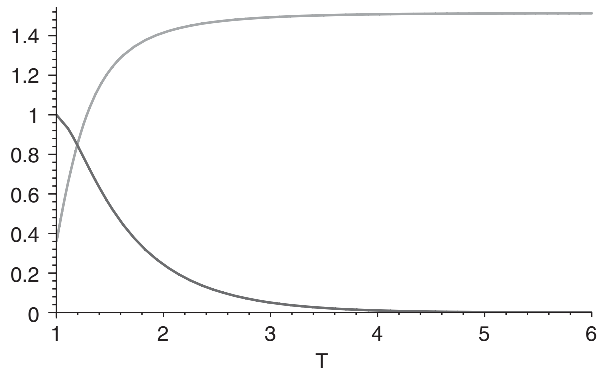

Fig. 1. Plot of $p_{t,T}$, $Y_{t,T}$ in terms of maturity T for $t=1$ and $x=1/(2\sqrt{2})$.

3.2. Exponential frequency

Suppose now that there exists a function $\psi(t)$ such that the functions $m(t)$, $\sigma(t)$ of (2), (6) admit the following representation:

$$\sigma(t)=\sqrt{2}\psi^2 e^{-\int^t \psi^2}, \quad m(t)=\psi^2-\dot{\psi}/\psi. \tag{37}$$

Then with an appropriate choice of integration constants we have

$$M(t)=-\log(\sqrt{2}\psi)+\int^t \psi^2(s)\,ds, \quad t^* \equiv \int^t \psi^2(s)\,ds, \quad x^*=\sqrt{\frac{2x}{\psi^2}e^{\int^t \psi^2}} \tag{38}$$

and hence from (14) we find that the frequency decreases exponentially: $\omega^*(t^*)=e^{-t^*}$. Even though Eq. (18) has a time-dependent frequency, the solutions of Eq. (18) can be obtained in an explicit way. Skipping details we have

$$\begin{aligned}\lambda_1^*(T^*|t^*)&=e^{-t^*}\left(K_1(e^{-t^*})I_0(e^{-T^*})+I_1(e^{-t^*})K_0(e^{-T^*})\right),\\ \lambda_2^*(T^*|t^*)&=I_0(e^{-t^*})K_0(e^{-T^*})-K_0(e^{-t^*})I_0(e^{-T^*}),\end{aligned} \tag{39}$$

where $I_n(z)$, $K_n(z)$, $n=0,1$ are modified Bessel functions of first and second kind. The value of claims follows from (25). In particular, (30) gives that the price of bonds is given by

$$p_{t,T}(x)=e^{(1/2)\varphi_1(T)}(B(T|t))^{-1/2}\exp\left[-\frac{e^{\varphi_1(t)}}{\psi^2(t)}\frac{A(T|t)}{B(T|t)}x\right], \tag{40}$$

where

$$\begin{aligned}A(T|t)&\equiv K_1(e^{-T^*})I_1(e^{-t^*})-I_1(e^{-T^*})K_1(e^{-t^*}),\\ B(T|t)&\equiv I_0(e^{-t^*})K_1(e^{-T^*})+K_0(e^{-t^*})I_1(e^{-T^*}).\end{aligned} \tag{41}$$

Note that, as expected, p decreases with the maturity time from the initial value $p_{t,t}=1$ to the limit value

$$p_{t,T}(x)\underset{T\to\infty}{\to}\frac{1}{\sqrt{I_0(e^{-t^*})}}\exp\left[-\frac{e^{\varphi_1(t)}}{\psi^2(t)}\frac{I_1(e^{-t^*})}{I_0(e^{-t^*})}x\right]>0$$

which is strictly positive, unlike what happens in the CIR model and former case (see (35)).

In Fig. 2 we plot both $p_{t,T}$ and $Y_{t,T}$ in terms of maturity T for $t=0$ and $x=\frac{2}{3}$ with the election $\psi(t)=1$. Note that the total decrease in the price of the bond is only around 30% of its initial value. In this case, the yield curve is always inverted and tends to zero, i.e., longer-term investments yield lower returns. To explain such a bizarre situation on intuitive grounds we note that in this model the interest rate has the

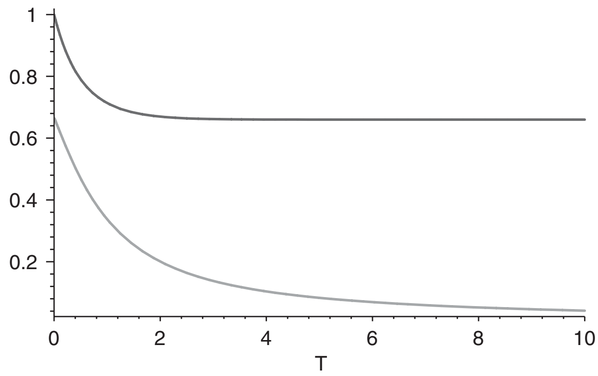

Fig. 2. Plot of $p_{t,T},\ Y_{t,T}$ in terms of maturity T for $t=0,\ x=2$.

representation $r_t = R_t^2$ where R_t is normally distributed as

$$R_t \sim \mathcal{N}\left(\sqrt{2}c\mathrm{e}^{-t}, 2t\mathrm{e}^{-2t}\right).$$

Thus $R_t \to 0$ as $t \to \infty$ with probability one and so it does the interest rate. A vanishing interest rate corresponds to bond prices tending to a constant close to one but to vanishing yields.

3.3. Valuation in the classical CIR model

Suppose next that the temporal evolution of the interest rate r_t is defined by (6) with constant parameters σ and m. This is the classical CIR model where the payoff is arbitrary. In this case one can skip the task of solving (18) by directly considering Eq. (28), which, unlike (18), is then a *trivial constant-coefficients linear* ODE. Solutions satisfying the initial conditions (28.2) are given by

$$\lambda_1(T|t) = \frac{\sigma^2(t)\mathrm{e}^{2M(t)}}{\omega_- - \omega_+}(\omega_-\mathrm{e}^{\omega_+(T-t)} - \omega_+\mathrm{e}^{\omega_-(T-t)}),$$
$$\lambda_2(T|t) = \frac{\sigma^2(t)\mathrm{e}^{2M(t)}}{\omega_- - \omega_+}(\mathrm{e}^{\omega_-(T-t)} - \mathrm{e}^{\omega_+(T-t)}), \tag{42}$$

where $\omega_\pm = m \pm \eta/2$ and $\eta \equiv \sqrt{4m^2 + 2\sigma^2}$.

Substituting this into (25) and changing variables via $X \equiv \mathrm{e}^{-2M(T)}X^{*2}/4$ we obtain the t-price $v_{t,T}(x)$ of a security that pays $\Theta(r_T)$ at maturity as

$$v_{t,T}(x) = \int \sqrt{D}\,\exp\left[-A(\sqrt{X} - C\sqrt{x})^2 - Bx\right]\Theta(X)\,\mathrm{d}X, \tag{43}$$

where $\xi \equiv \mathrm{e}^{(T-t)}$ and

$$A \equiv \frac{2}{\sigma^2 B}, \quad B \equiv \frac{2(\xi^\eta - 1)}{(2m+\eta)(\xi^\eta - 1) + 2\eta},$$
$$C \equiv \frac{2\eta\xi^{\eta/2}}{(2m+\eta)(\xi^\eta - 1) + 2\eta}, \quad D \equiv \frac{\eta}{2\pi}\frac{\xi^{m+\eta/2}}{\xi^\eta - 1}. \tag{44}$$

Formula (43) solves in an *explicit way* the problem of valuation of general securities under the CIR model, and hence represents, to our knowledge, a novel important result.

In particular, if $\Theta = 1$ we recover the result of Cox et al. [2] for the price of a bond:

$$p_{t,T}(x) = \sqrt{C}\xi^{m/2}\mathrm{e}^{-Bx}. \tag{45}$$

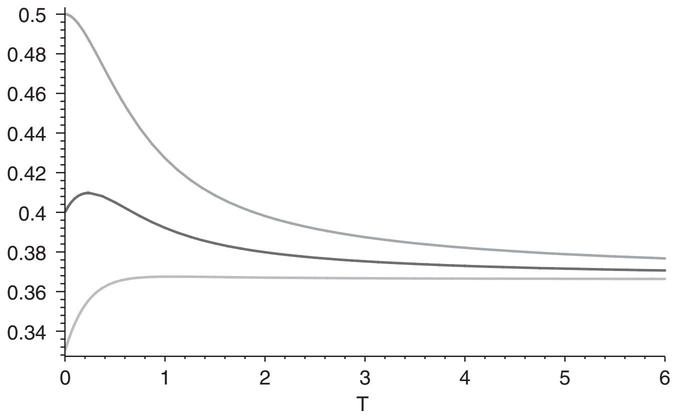

Fig. 3. The yield curves for $x = 0.33, 0.4, 0.5$ and parameters $t = 0,\ m = 1,\ \sigma = 2$.

It follows that $p_{t,T}$ decays to 0 exponentially with the maturity time:

$$p_{t,T} \approx \sqrt{\frac{2\eta}{\eta + 2m}} \mathrm{e}^{-(\eta-2m)(T-t)/4} \underset{T\to\infty}{\to} 0. \tag{46}$$

Using (45) an easy calculation shows that as $T \to \infty$ the yield tends to a fixed quantity independent of the starting value r_t:

$$\lim_{T\to\infty} Y_{t,T}(r_t) = \eta/4 - m/2 \equiv \frac{m}{2}\left(\sqrt{1 + \frac{\sigma^2}{2m^2}} - 1\right). \tag{47}$$

Note that the yield to perpetuity increases with σ but decreases with the parameter m. This is what is to be expected on intuitive grounds as σ is proportional to the infinitesimal volatility while m is related to the mean reversion level of the interest rate (see (6)). Depending in values of the relevant parameters the yield curve may be normal, inverted or humped. In Fig. 3 we plot yield curves for $x = 0.33$, $x = 0.4$ and $x = 0.5$ and parameter values $t = 0,\ m = 1$ and $\sigma = 2$ that show this. Note how the direction of the slope is quite sensitive to the original value $x \equiv r_t$. This behavior of the yield curve can be, again, partially explained in terms of the process R_t such that $r_t = R_t^2/4$. Here R_t is an Ornstein–Uhlenbeck process, having normal distribution with mean and variance

$$\mu_t = c\mathrm{e}^{-mt}, \quad v_t = \frac{\sigma^2}{2m}(1 - \mathrm{e}^{-2mt}), \quad c \equiv 2\sqrt{r_0}.$$

Thus, the mean value of the interest rate

$$\bar{r}_t = \frac{1}{4}\left(\frac{\sigma^2}{2m} + \left(4r_0 - \frac{\sigma^2}{2m}\right)\mathrm{e}^{-2mt}\right)$$

decreases or increases in time depending on whether or not the condition $r_0 > \sigma^2/8m$ holds.

Actually, this latter condition determines roughly whether the yield curve increases or decreases. This is to be expected since if $\bar{r}_t$ is increasing, say, towards a limit value $\sigma^2/8m$ one may expect that the longer the maturity the higher the yield and hence that yield curves also increase.

4. Conclusions

In this paper we have considered the problem of how to prize general European securities in the time-dependent CIR model, i.e., whenever the instantaneous interest rate satisfies a time-dependent Feller SDE as suggested originally in Ref. [11]. We show how to solve explicitly this SDE whenever a certain condition is satisfied and give the temporal evolution of the interest rate. Motivated by the method of solution we find a coordinate transformation that converts the prizing PDE (5) in a heat equation with a time-dependent harmonic oscillator potential. We next reduce the former to solving the *linear ordinary differential equation of the classical harmonic oscillator*.

We also find several reductions under which this ODE is solvable in closed form. In these cases, explicit formulae for prizing are given and the behavior of bonds is discussed. In particular we have given the solution to the problem of valuation of general securities under the CIR model in an explicit way. For all these models the yield chart is given and its behavior discussed.

Acknowledgments

This work was partially sponsored by CMICYT BFM2002-02609 and SA078/03.

References

[1] I. Karatzas, S.E. Shreve, Brownian Motion and Stochastic Calculus, Springer, Berlin, 1991.
[2] J.C. Cox, J.E. Ingersoll, S.A. Ross, Econometrica 53 (2) (1985) 385.
[3] O. Vasicek, J. Fin. Econ. 5 (1977) 177–188.
[4] S. Galluzio, Physica A 269 (1999) 61.
[5] H.-P. Bermin, A. Higa, J. Perelló, Physica A 335 (2005) 152.
[6] U. Jaekel, Physica A 352 (2005) 584.
[7] W. Feller, Ann. Math. 54 (1) (1955) 175.
[8] V. Safonov, J. Appl. Phys. 85 (8) (1999) 4370.
[9] Publications of the Department of Finance, Canada, Debt Management Report 2002–2003, available on-line at ⟨http://www.fin.gc.ca/toce/2003/dmr03_e.html⟩. See also Publications of the Department of Finance, Canada, Affine term-structure models available on-line at ⟨http://www.bankofcanada.ca/en/res/wp/2001/wp01-15a.pdf⟩.
[10] Publications of the Danmarks Nationalbank, Danish Government Borrowing and Debt 2005, available on-line at ⟨http://www.nationalbanken.dk/C1256BE9004F6416/side/⟩.
[11] J. Hull, A. White, Rev. Fin. Stud. 3 (4) (1990) 573.
[12] D. Duffie, Dynamic Asset Pricing Theory, Princeton University Press, Princeton, NJ, 1996.
[13] D. Duffie, R. Kan, Math. Finance 6 (4) (1996) 379.
[14] D. Brigo, F. Mercurio, Finance Stochastics 5 (3) (2001) 369.
[15] D. Brigo, F. Mercurio, Interest Rate Models: Theory and Practice, Springer, Berlin, 2001.
[16] R.N. Mantegna, H.E. Stanley, An Introduction to Econophysics, Cambridge University Press, Cambridge, 2000.
[17] J. Voit, The Statistical Mechanics of Financial Markets, Springer, Berlin, 2003.
[18] H. Kleinert, Path Integrals in Quantum Mechanics Statistics, Polymer Physics, and Financial Markets, World Scientific, Singapore, 2001.
[19] J. Bouchard, P. Potters, Theory of Financial Risk, from Statistical Physics to Risk Management, Cambridge University Press, Cambridge, 2000.
[20] G. Vasconzelos, Braz. J. Phys. 34 (3B) (2004) 1039.
[21] T. Mikosch, Elementary Stochastic Calculus with Finance in View, World Scientific, Singapore, 1998.
[22] J.E. Ingersoll, Theory of Financial Decision Making, Rowman & Littlefield, Savage, MD, 1987.
[23] J. Hull, Options, Futures, Derivatives, Prentice Hall University Press, 1997.
[24] S. Svoboda, Interest Rate Modelling (Finance & Capital Markets Series), Palgrave Macmillan, New York, 2003.
[25] ⟨http://www.federalreserve.gov/releases/h15⟩.
[26] ⟨http://finance.yahoo.com/q/bc?s=⟩.
[27] ⟨http://www.fincad.com/default.asp?id=20700s=Productsn=Models⟩.
[28] R.N. Mantegna, H.E. Stanley, Nature 376 (1995) 46.
[29] J. Perelló, J. Masoliver, Physica A 314 (2002) 736.
[30] S. Galluzio, Eur. Phys. J. B 20 (4) (2001) 595.
[31] J. Perelló, J. Masoliver, Physica A 308 (2002) 420.
[32] S. Galluzio, G. Caldarelli, M. Marsilli, Y.-C. Zhang, Physica A 245 (1997) 423.
[33] J. Villarroel, J. Comput. Appl. Math. 158 (1) (2003) 225.
[34] B. Fuller, Mathematics of Classical and Quantum Physics, Addison-Wesley, New York, 1970.
[35] J. Villarroel, Stochastic Anal. Appl. 21 (6) (2003) 1391.
[36] J. Villarroel, M.J. Ablowitz, Stud. Appl. Math. 109 (2002) 151.
[37] J. Villarroel, M.J. Ablowitz, Nonlinearity 17 (2004) 1843–1866.