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Factorization of a class of almost linear second-order differential equations

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Abstract

A general type of almost linear second-order differential equations, which are directly related to several interesting physical problems, is characterized. The solutions of these equations are obtained using the factorization technique, and their non-autonomous invariants are also found by means of scale transformations.

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1. Introduction

From previous works [1–3], we know that certain nonlinear second-order ordinary differential equations (ODE) can be factorized under some conditions that coincide with those of integrability obtained from a Painlevé analysis. These kinds of equations frequently appear when looking for travelling wave solutions of interesting nonlinear physical equations such as Korteweg–de Vries–Burgers [1, 2], Kadomtsev–Petviashvili [1], and Benjamin–Bona–Mahony equations [3]. We also know that the factorizations are directly related to first integrals of the equation that are a kind of Bohlin's first integrals [1, 4].

In this work we deal with a class of second-order differential equations with variable coefficients that will be called 'almost linear' (AL). In general, there are no standard techniques to solve this kind of nonlinear equations. Here, our aim is to solve them by using the factorization technique appropriately adapted. Then, we want to find the first integrals of this class of equations, which are related to the factors entering the factorization. We note that this class includes the generalized Emden–Fowler equation, with a second-order nonlinearity in the dependent variable [5–8], which appears in various fields of physics.

In section 2, we introduce AL second-order ODE which will be transformed into canonical forms. Then, in section 3, one type of the canonical equations is factorized in a standard way.

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In section 4, we will study the factorization of the AL second-order ODE of section 2. In principle it is difficult to factorize them directly, but we can achieve it by using the factorization of their canonical forms. We will consider again a scale transformation to get the invariants (first integrals) of these general equations in section 5. Finally, in the last section we will draw some conclusions.

2. The general form of the almost linear second-order ordinary differential equation

Let us consider the second-order ODE with variable coefficients of the form

$$\frac{\mathrm{d}^2Y(t)}{\mathrm{d}t^2}+F_1(t)\frac{\mathrm{d}Y(t)}{\mathrm{d}t}+F_2(t)Y^2(t)+F_3(t)Y(t)=0, \tag{2.1}$$

where the coefficients $F_1(t)$, $F_2(t)\neq 0$ and $F_3(t)$ are functions of the independent variable t. This equation will be called in the following 'almost linear' second-order ODE because it is linear, except for the presence of the term in $Y^2(t)$. Equation (2.1) can be set in a canonical form by introducing the following scale transformation (see [9]):

$$Y(t)=\lambda(t)(W(z)-b_0)\qquad \mathrm{d}z=\phi(t)\,\mathrm{d}t \tag{2.2}$$

that changes (2.1) into

$$\begin{aligned}\frac{\mathrm{d}^2W(z)}{\mathrm{d}z^2}&+\frac{1}{\phi}\left(\frac{1}{\phi}\frac{\mathrm{d}\phi}{\mathrm{d}t}+\frac{2}{\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}t}+F_1\right)\frac{\mathrm{d}W(z)}{\mathrm{d}z}+F_2\frac{\lambda}{\phi^2}W(z)^2\\&+\frac{1}{\phi^2}\left(\frac{1}{\lambda}\frac{\mathrm{d}^2\lambda}{\mathrm{d}t^2}+\frac{F_1}{\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}t}-2b_0F_2\lambda+F_3\right)W(z)\\&-\frac{b_0}{\phi^2}\left(\frac{1}{\lambda}\frac{\mathrm{d}^2\lambda}{\mathrm{d}t^2}+\frac{F_1}{\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}t}-b_0F_2\lambda+F_3\right)=0,\end{aligned} \tag{2.3}$$

where $\lambda(t)$, $\phi(t)$ and b_0 should be selected in order to write (2.3) as one of the Painlevé classifications of second-order differential equations [10]. As is well known [9], there are two independent canonical forms for equation (2.3)

$$\frac{\mathrm{d}^2W(z)}{\mathrm{d}z^2}-6W^2(z)+6b_0^2=0 \tag{2.4}$$

and

$$\frac{\mathrm{d}^2W(z)}{\mathrm{d}z^2}-6W^2(z)-z=0. \tag{2.5}$$

Equation (2.5) is the first Painlevé transcendent which is not factorizable. Therefore, in the following we are only interested in the equations of the form (2.1) that can be transformed into (2.4). By comparison of (2.3) and (2.4), $\lambda(t)$ and $\phi(t)$ must be defined as

$$\lambda(t)=F_2^{-1/5}(t)\,\mathrm{e}^{-\frac{2}{5}\int F_1(t)\mathrm{d}t}\qquad \phi(t)^2=-\frac{\lambda(t)F_2(t)}{6} \tag{2.6}$$

and the functions F_1, F_2 and F_3 should satisfy the relation

$$F_3(t)=2b_0F_2(t)\lambda(t)-\frac{1}{\lambda(t)}\frac{\mathrm{d}^2\lambda(t)}{\mathrm{d}t^2}-\frac{F_1(t)}{\lambda(t)}\frac{\mathrm{d}\lambda(t)}{\mathrm{d}t}. \tag{2.7}$$

In conclusion, we have shown that (2.1) can be transformed into the canonical equation (2.4) if and only if the functions F_1, F_2 and F_3 are related through condition (2.7).

Provided that condition (2.7) is fulfilled, the reduction of (2.1) to the canonical form (2.4) presents several advantages that we shall explore in the following. More precisely, the canonical form (2.4) is rather easy to factorize. This factorization allows us to perform the inverse of the scale transformation (2.2) to obtain a factorization for equation (2.1) which, in principle, is quite difficult to factorize directly.

3. Factorization of the canonical equation

Let us consider equation (2.4) with constant coefficients, it can be factorized in the form

$$\left(\frac{\mathrm{d}}{\mathrm{d}z} - f_2(W)\right)\left(\frac{\mathrm{d}}{\mathrm{d}z} - f_1(W)\right) W = 0, \tag{3.1}$$

where $f_1(W)$ and $f_2(W)$ satisfy the following relations:

$$f_1 f_2 W + 6W^2 - 6b_0^2 = 0 \tag{3.2}$$

$$f_2 = -W\frac{\partial f_1}{\partial W} - f_1. \tag{3.3}$$

After substituting (3.3) into (3.2) we get for f_1 the consistency equation

$$W f_1 \frac{\partial(f_1)}{\partial W} + f_1^2 - 6W + 6\frac{b_0^2}{W} = 0, \tag{3.4}$$

whose solutions are

$$f_1(W) = \pm\sqrt{4W - 12\frac{b_0^2}{W} - \frac{c}{W^2}}, \tag{3.5}$$

where c is an integration constant. Now, using these functions in (3.1) we obtain

$$\frac{\mathrm{d}W}{\mathrm{d}z} \mp \sqrt{4W^3 - 12b_0^2 W - c} = 0. \tag{3.6}$$

It is easy to see [11] that the solutions of (3.6) are given in terms of elliptic functions,

$$W(z) = \wp(z + z_0; g_2, g_3), \tag{3.7}$$

where $\wp(z + z_0; g_2, g_3)$ is the Weierstrass function, z_0 is a half-period, and the invariants are

$$g_2 = 12b_0^2 \qquad g_3 = c. \tag{3.8}$$

The degenerate cases in which the discriminant $\Delta = g_2^3 - 27g_3^2 = 27\left(8b_0^3 - c\right)\left(8b_0^3 + c\right)$ of the Weierstrass function is zero [11, 12] supply the particular solutions:

- $c = -8b_0^3$

$$W(z) = -b_0 + 3b_0 \cos^{-2}[\sqrt{3b_0}(z + z_0)] \tag{3.9}$$

- $c = 8b_0^3$

$$W(z) = b_0 - 3b_0 \cosh^{-2}[\sqrt{3b_0}(z + z_0)]. \tag{3.10}$$

Now, we can use the above results to find the factorization and solutions of the AL second-order ODE with variable coefficients (2.1) by using the transformation (2.2).

4. Factorization of the almost linear second-order ODE

If equation (2.1) admits a general factorization of the form

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - R_2(Y,t)\right)\left(\frac{\mathrm{d}}{\mathrm{d}t} - R_1(Y,t)\right)Y = 0 \tag{4.1}$$

then, $R_1(Y,t)$ and $R_2(Y,t)$ should satisfy the following consistency conditions:

$$R_1(Y,t)R_2(Y,t) - \frac{\partial R_1(Y,t)}{\partial t} - F_2(t)Y - F_3(t) = 0 \tag{4.2}$$

$$R_1(Y,t) + R_2(Y,t) + Y\frac{\partial R_1(Y,t)}{\partial Y} + F_1(t) = 0. \tag{4.3}$$

The elimination of $R_2(Y,t)$ between (4.2) and (4.3) provides for $R_1(Y,t)$ the nonlinear partial differential equation

$$\frac{\partial R_1(Y,t)}{\partial t} + YR_1(Y,t)\frac{\partial R_1(Y,t)}{\partial Y} + R_1(Y,t)^2 + F_1(t)R_1(Y,t) + F_2(t)Y + F_3(t) = 0. \tag{4.4}$$

Obviously, it is very difficult to determine R_1 from this equation. Nevertheless, we can find R_1 and R_2 by using the factorization of the canonical equation mentioned above (in section 3). We consider the first-order ODE

$$\left[\frac{\mathrm{d}}{\mathrm{d}z} - f_1(W)\right]W(z) = 0, \tag{4.5}$$

where f_1 was given by (3.5) and $W(z)$ can be written from (2.2) as

$$W(z) = b_0 + \frac{Y(t)}{\lambda(t)} \qquad \mathrm{d}z = \phi(t)\,\mathrm{d}t. \tag{4.6}$$

Then, substituting (4.6) into (4.5) and using the form of $\phi(t)$ given by (2.6), we get another first-order ODE but in terms of Y and t,

$$\left[\frac{\mathrm{d}}{\mathrm{d}t} - \left(\frac{1}{\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}t} + \sqrt{\frac{F_2}{6}Q(Y,t)}\right)\right]Y(t) = 0, \tag{4.7}$$

where

$$Q(Y,t) = -4Y - 12b_0\lambda + \left(c + 8b_0^3\right)\frac{\lambda^3}{Y^2}. \tag{4.8}$$

Now, comparing (4.7) with (4.1), we have

$$R_1(Y,t) = \frac{1}{\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}t} + \sqrt{\frac{F_2}{6}Q(Y,t)} \tag{4.9}$$

and from (4.3) we have also

$$R_2(Y,t) = -F_1 - \frac{1}{\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}t} + (Y + 2b_0\lambda)\sqrt{\frac{6F_2}{Q(Y,t)}}. \tag{4.10}$$

It is a trivial exercise to prove that (4.9) and (4.10) satisfy (4.2) and (4.3) if and only if F_3 satisfy (2.7). Therefore, we can say that equation (2.1) can be factorized as shown by (4.1), just in the case in which it can be transformed into the canonical form (2.4).

5. Invariants of the canonical and the almost linear second-order ODE

An additional bonus of transformation (2.2) is that it provides us with invariants for equation (2.1) by identifying the invariants of (2.4). Actually, (3.6) can be trivially rewritten as

$$-c = \left(\frac{\mathrm{d}W}{\mathrm{d}z}\right)^2 - \left(4W^3 - 12b_0^2 W\right) \tag{5.1}$$

and easily factorized as the product

$$-c = \left(\frac{\mathrm{d}W(z)}{\mathrm{d}z} - H(W)\right)\left(\frac{\mathrm{d}W(z)}{\mathrm{d}z} + H(W)\right), \tag{5.2}$$

where

$$H(W) = \sqrt{4W^3 - 12b_0^2 W}. \tag{5.3}$$

Equation (5.2) can also be expressed as

$$-c = I_1 I_2, \tag{5.4}$$

where I_1, I_2 are invariants depending also on the independent variable z, related to the Bohlin's first integrals [1, 4], defined by

$$I_1 = \left(\frac{\mathrm{d}W}{\mathrm{d}z} - H(W)\right)\exp\left\{\int\left[\frac{6\left(W(z)^2 - b_0^2\right)}{H(W)}\right]\mathrm{d}z\right\} \tag{5.5}$$

and

$$I_2 = \left(\frac{\mathrm{d}W}{\mathrm{d}z} + H(W)\right)\exp\left\{\int\left[-\frac{6\left(W(z)^2 - b_0^2\right)}{H(W)}\right]\mathrm{d}z\right\}. \tag{5.6}$$

It is a rather easy task to check that, with these definitions I_1 and I_2 are constants of motion for (2.4), i.e., $\mathrm{d}I_j/\mathrm{d}z = 0$, $j = 1, 2$.

These invariants also give rise, through the scale transformation (2.2), to non-autonomous invariants of (2.1) in terms of Y and t. Actually, if we perform this transformation in (5.1) and (5.2), we obtain

$$I_1 = \sqrt{\frac{-6}{F_2\lambda^3}}\left[\frac{\mathrm{d}Y}{\mathrm{d}t} - \frac{1}{\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}t}Y - \sqrt{\frac{-F_2}{6}}P\right]\exp\left\{\int\left[(Y^2 + 2b_0\lambda Y)\frac{\sqrt{-6F_2}}{P}\right]\mathrm{d}t\right\} \tag{5.7}$$

$$I_2 = \sqrt{\frac{-6}{F_2\lambda^3}}\left[\frac{\mathrm{d}Y}{\mathrm{d}t} - \frac{1}{\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}t}Y + \sqrt{\frac{-F_2}{6}}P\right]\exp\left\{\int\left[-(Y^2 + 2b_0\lambda Y)\frac{\sqrt{-6F_2}}{P}\right]\mathrm{d}t\right\}, \tag{5.8}$$

where

$$P(Y, t) = \sqrt{4Y^3 + 12b_0\lambda Y - 8b_0^3\lambda^3}. \tag{5.9}$$

A direct calculation allows us to check again that I_1, I_2 as defined in (5.7) and (5.8) are constants of motion for (2.1), $\mathrm{d}I_j/\mathrm{d}t = 0$, $j = 1, 2$, if and only if F_3 satisfy (2.7).

6. Conclusions

In this paper, we have factorized both a class of general AL second-order ODE with variable coefficients and the transformed canonical equation which is a second-order nonlinear ODE with constant coefficients. We have shown that the relation between these two factorizations imposes the restriction (2.7) on the variable coefficients of the initial class of equations, and how they provide us with the solutions of these equations. Then, the non-autonomous invariants (first integrals related to Bohlin's first integrals [4]) of these equations that are also directly related to the factorizations have been obtained by using scale transformations. Both methods impose the same restriction (2.7) on the coefficients of the equation. This restriction coincides with that obtained by using the Painlevé criteria of integrability. Finally, we mention that the results obtained here were checked by the MAPLE symbolic program.

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