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Non-isospectral scattering problems: Painlevé truncation for hierarchies

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Abstract

In a series of recent papers we have used partial differential equations (PDEs) of a certain form, having non-isospectral scattering problems, in the construction of hierarchies of integrable PDEs together with their underlying linear problems. Here we show how these ideas provide a key to obtaining truncation results for entire hierarchies of integrable PDEs. This new method allows us to obtain, in a very straightforward way, results first obtained by Weiss. The approach given here is readily extended to the various generalizations of the truncation process that appear in the literature.
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1. Introduction

Integrable systems, since the introduction of the inverse scattering transform [1], have developed to form an important area of research that encompasses aspects of a wide range of disciplines in mathematics and physics: from mathematical analysis to theoretical physics to algebraic geometry and beyond. The key to integrability remains of course the solution of a system through the use of an underlying linear problem. This concept today has come to include linear problems for partial differential equations (PDEs) in mul-

tidimensions, for ordinary differential equations, for differential-difference equations, for discrete systems, and for lattice systems.

In our recent work [2–5], we have been interested in one particular kind of PDE in multidimensions, namely, those associated to non-isospectral scattering problems, i.e., where the parameter in the scattering problem is no longer constant but satisfies certain differential constraints [6–8]; we also discussed reductions to non-isospectral scattering problems in $1 + 1$ dimensions, previous studies of which can be found, for example, in [9,10]. We gave in [3] several reasons for our interest, amongst them the information that such scattering problems could give about the Painlevé analysis, including truncation, of whole hierarchies of PDEs. This last is important because the truncation of

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Painlevé expansions, being a method of obtaining Lax pairs for integrable PDEs, is fundamentally related to the question of underlying linear problems and so to integrability, and it is a natural question to ask how the truncation process can be extended from particular PDEs to their hierarchies. Indeed, this question was looked at by Weiss who, amongst other examples, extended his results on truncation for the KdV equation [11,12] and for the fifth order KdV equation [11] to truncation for the entire KdV hierarchy [12].

The aim of the present Letter is to explain our comment in [3], and to show how non-isospectral scattering problems can be used to obtain results on truncation for whole hierarchies of equations. As an example we will show how to recover the above-mentioned results of Weiss for the KdV hierarchy. Our claim is that our approach is much simpler than that employed by Weiss, in that it does not depend on knowledge of a modified hierarchy or of a Schwarzian formulation, although this last is readily recovered. The idea explained here, once understood, is readily extended to the various generalizations of the truncation process, e.g., those in [13,14].

2. KdV hierarchy: a non-isospectral construction

In [3] we observed that instead of considering the KdV hierarchy in the form

$$U_{t_{2n+1}} = \mathcal{R}^n U_x, \quad n = 0, 1, 2, \ldots, \tag{1}$$

where

$$\mathcal{R} = \partial_x^2 + 4U + 2U_x \partial_x^{-1}, \quad \partial_x = \partial/\partial x, \tag{2}$$

is the usual KdV recursion operator, we could instead consider iterations of equations in $U(x, \tau, t)$ of the form

$$U_t = \mathcal{R} U_\tau. \tag{3}$$

For example, by setting first $t = t_5$ and $\tau = t_3$, and then setting $t = t_3$ and taking the reduction $\partial/\partial \tau = \partial/\partial x$,

$$U_{t_5} = \mathcal{R} U_{t_3}, \qquad U_{t_3} = \mathcal{R} U_x, \tag{4}$$

we obtain the fifth order KdV equation $U_{t_5} = \mathcal{R}^2 U_x$. Similarly, iterations of associated linear problems lead to linear problems for members of the KdV hierarchy.

Eq. (3) has the non-isospectral scattering problem

$$\psi_{xx} + (U - \lambda)\psi = 0,$$
$$\psi_t = 4\lambda \psi_\tau + 2[\partial_x^{-1} U_\tau]\psi_x - U_\tau \psi, \tag{5}$$

where $\lambda = \lambda(\tau, t)$ satisfies

$$\lambda_t = 4\lambda \lambda_\tau, \tag{6}$$

and so, for example, the same iteration as before, i.e.,

$$\psi_{t_5} = 4\lambda \psi_{t_3} + 2[\partial_x^{-1} U_{t_3}]\psi_x - U_{t_3}\psi,$$
$$\lambda_{t_5} = 4\lambda \lambda_{t_3}, \tag{7}$$

and

$$\psi_{t_3} = (4\lambda + 2U)\psi_x - U_x\psi,$$
$$\lambda_{t_3} = 4\lambda \lambda_x = 0, \tag{8}$$

leads to the temporal half of the Lax pair for the fifth order KdV equation,

$$\psi_{t_5} = \left[16\lambda^2 + 8\lambda U + 2U_{xx} + 6U^2\right]\psi_x$$
$$\qquad - [4\lambda U_x + U_{xxx} + 6U U_x]\psi, \qquad \lambda_{t_5} = 0. \tag{9}$$

Clearly, if instead of taking $t = t_3$ and the reduction $\partial/\partial \tau = \partial/\partial x$, we had taken $t = t_3$ and $\tau = y$, then we would have obtained $U_{t_5} = \mathcal{R}^2 U_y$ together with its non-isospectral scattering problem. That is, instead of iterating to obtain the (isospectral) KdV hierarchy, we could also iterate (see [3]) to obtain the non-isospectral hierarchy $U_{t_{2n+1}} = \mathcal{R}^n U_y$.

Our strategy to obtain truncation results for the KdV hierarchy is therefore to consider the truncation for Eq. (3), as well as for the reduction of (3) that gives the KdV equation $U_{t_3} = \mathcal{R} U_x$, and then to consider the iteration of these results.

3. Truncation for a non-isospectral equation

In this section we give results for the truncation of Eq. (3). In fact we will for the sake of completeness give results for the slightly more general equation[1]

$$u_{xt} = u_{xxx\tau} + 4u_x u_{x\tau} + 2u_{xx} u_\tau + g(\tau, t); \tag{10}$$

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[1] If we replace $g(\tau, t)$ here by $f(x, \tau, t)$, the Painlevé test requires $f_x = 0$.

setting $g(\tau, t) = 0$ and $u_x = U$ in this last yields (3). The truncation then proceeds analogously to that of the generalization of the $2 + 1$ classical Boussinesq system (128), (129) of [14] (see Remark, p. 1913), with the exception that the higher order truncation discussed in [14] is not needed. What is then precisely the standard Weiss truncation [11,12] tells us (see [14] and references therein for notation): u given by

$$u = 2\chi^{-1} + v \tag{11}$$

satisfies (10) where the coefficients S, C^t and C^τ of the Riccati system

$$\chi_x = 1 + \frac{1}{2}S\chi^2, \tag{12}$$

$$\chi_t = -C^t + C^t_x \chi - \frac{1}{2}\big(C^t_{xx} + C^t S\big)\chi^2, \tag{13}$$

$$\chi_\tau = -C^\tau + C^\tau_x \chi - \frac{1}{2}\big(C^\tau_{xx} + C^\tau S\big)\chi^2, \tag{14}$$

are given by

$$S = 2v_x - 2\lambda, \qquad C^t = 4\lambda C^\tau - 2v_\tau, \tag{15}$$

$\lambda(\tau, t)$ is a function of integration, $g = \lambda_t - 4\lambda\lambda_\tau$ and v also satisfies (10).

In the notation of Weiss [11,12], here S is the Schwarzian derivative of φ, the function which defines the singular manifold,

$$S = \left(\frac{\varphi_{xx}}{\varphi_x}\right)_x - \frac{1}{2}\left(\frac{\varphi_{xx}}{\varphi_x}\right)^2,$$

$$\chi^{-1} = \left(\frac{\varphi_x}{\varphi} - \frac{1}{2}\frac{\varphi_{xx}}{\varphi_x}\right),$$

$$C^t = -\frac{\varphi_t}{\varphi_x}, \qquad C^\tau = -\frac{\varphi_\tau}{\varphi_x}. \tag{16}$$

Recognising that we have a relation of the form $C^t = \Gamma C^\tau + \widetilde{C}$ where $\Gamma_x = 0$, we see that instead of the Riccati system (12)–(14) we can consider the Riccati system [14]

$$\chi_x = 1 + \frac{1}{2}S\chi^2,$$

$$\chi_t = \Gamma\chi_\tau - \widetilde{C} + \widetilde{C}_x\chi - \frac{1}{2}(\widetilde{C}_{xx} + \widetilde{C}S)\chi^2, \tag{17}$$

where $S = 2v_x - 2\lambda$, $\Gamma = 4\lambda$ and $\widetilde{C} = -2v_\tau$. Imposing now the additional condition $g = 0$, we thus obtain, using the linearization $\chi^{-1} = \psi_x/\psi$ of (17), the Lax pair and non-isospectral condition given in

(5) and (6). The truncation (11) is then the Darboux transformation [15] for the non-isospectral Eq. (3). We note that Eq. (3) was in fact the first example given of an equation having a non-isospectral scattering problem [16].

4. Truncation for the KdV hierarchy

We now consider iterating the above results in order to find the truncation for the KdV hierarchy. We therefore need, as starting point for this iteration, the truncation for the KdV equation itself, $U_{t_3} = \mathcal{R}U_x$. This is readily obtained by making the reduction $\partial/\partial\tau = \partial/\partial x$ in the above results, and reads [11,12]

$$u = 2\chi^{-1} + v \tag{18}$$

where

$$\chi_x = 1 + \frac{1}{2}S\chi^2, \tag{19}$$

$$\chi_{t_3} = -C^{t_3} + C^{t_3}_x \chi - \frac{1}{2}\big(C^{t_3}_{xx} + C^{t_3}S\big)\chi^2, \tag{20}$$

and

$$S = 2v_x - 2\lambda, \qquad C^{t_3} = -4\lambda - 2v_x, \tag{21}$$

$V = v_x$ being a second solution of the KdV equation and λ now being constant.

Thus we obtain what is referred to as the singular manifold equation [11,12]—see [14] for an updated definition of this concept—for the KdV equation,[2]

$$C^{t_3} + S + 6\lambda = 0. \tag{22}$$

We now turn to the iteration process. We note that, since (17) is equivalent to the Lax pair given in (5), and since we obtain the same non-isospectral condition (6), this iteration is equivalent to that explained earlier for constructing the t_{2n+1} flow of the KdV hierarchy together with its (isospectral) Lax pair.

That is, truncation for the KdV hierarchy gives the Darboux transformation

$$U = 2\big(\chi^{-1}\big)_x + V = 2(\log\psi)_{xx} + V, \tag{23}$$

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[2] The singular manifold equation for (10) is obtained by eliminating v between the equations in (15), which gives $S + 2\lambda + \partial_x \int^\tau (C^t - 4\lambda C^\tau)\,d\tau' = 0$.

together with the non-linearization under $\psi_x/\psi = \chi^{-1}$ of the Lax pair (Riccati pseudopotential) for the t_{2n+1} flow. It only remains therefore to consider the iteration of the singular manifold equations, in order to find the singular manifold equation for the t_{2n+1} flow.

In (15) we set $t = t_{2n+1}$ and $\tau = t_{2n-1}$, then $t = t_{2n-1}$ and $\tau = t_{2n-3}$, and iterate to obtain:

$$C^{t_{2n+1}} = 4\lambda C^{t_{2n-1}} - 2v_{t_{2n-1}} \tag{24}$$

$$= 4\lambda\left(4\lambda C^{t_{2n-3}} - 2v_{t_{2n-3}}\right) - 2v_{t_{2n-1}} \tag{25}$$

$$\vdots$$

$$= (4\lambda)^{n-1}C^{t_3} - 2\sum_{k=2}^{n}(4\lambda)^{n-k}v_{t_{2k-1}}. \tag{26}$$

We now note that since $\mathcal{R}^n U_x = \partial_x L_{n+1}[U]$, where $L_n[U]$ satisfies the Lenard recursion relation [17],

$$\partial_x L_{n+1}[U] = \left(\partial_x^3 + 4U\partial_x + 2U_x\right)L_n[U],$$

$$L_0[U] = \frac{1}{2}, \qquad L_1[U] = U, \tag{27}$$

we can replace $v_{t_{2k-1}}$ in the above by $L_k[V]$, and since $V = \lambda + \frac{1}{2}S$ we thus obtain

$$C^{t_{2n+1}} = (4\lambda)^{n-1}C^{t_3} - 2\sum_{k=2}^{n}(4\lambda)^{n-k}L_k\left[\lambda + \frac{1}{2}S\right] \tag{28}$$

$$= (4\lambda)^{n-1}(-6\lambda - S)$$
$$- 2\sum_{k=2}^{n}(4\lambda)^{n-k}L_k\left[\lambda + \frac{1}{2}S\right] \tag{29}$$

$$= -2(4\lambda)^n\frac{1}{2} - 2(4\lambda)^{n-1}\left(\lambda + \frac{1}{2}S\right)$$
$$- 2\sum_{k=2}^{n}(4\lambda)^{n-k}L_k\left[\lambda + \frac{1}{2}S\right] \tag{30}$$

$$= -2\sum_{k=0}^{n}(4\lambda)^{n-k}L_k\left[\lambda + \frac{1}{2}S\right]. \tag{31}$$

That is, the singular manifold equation for the t_{2n+1} flow of the KdV hierarchy is

$$C^{t_{2n+1}} + 2\sum_{k=0}^{n}(4\lambda)^{n-k}L_k\left[\lambda + \frac{1}{2}S\right] = 0. \tag{32}$$

This result is equivalent to that given by Weiss [12], although the above derivation is much simpler. We note that this derivation does not rely on the knowledge of a

modified hierarchy. Neither is Weiss's identification of S with the Schwarzian derivative of φ important here, although his Schwarzian formulation is easily recovered.

As an example of the above, we consider the case $n = 2$; the above result then tells us that the singular manifold equation for the fifth order KdV equation is

$$C^{t_5} + 2(4\lambda)^2 L_0\left[\lambda + \frac{1}{2}S\right] + 2(4\lambda)L_1\left[\lambda + \frac{1}{2}S\right]$$
$$+ 2L_2\left[\lambda + \frac{1}{2}S\right] = 0, \tag{33}$$

that is [11,12],

$$C^{t_5} + S_{xx} + \frac{3}{2}S^2 + 10\lambda S + 30\lambda^2 = 0. \tag{34}$$

5. Extensions of the above results

The main aim of this Letter has been to present a new method of obtaining truncation results for hierarchies of integrable PDEs. As an illustration of our approach we have recovered the results of Weiss for the KdV hierarchy [12]. Here we briefly note some extensions of these results for the KdV hierarchy to related hierarchies. Further applications of the approach developed here will be given in later papers.

Example one. It is straightforward to iterate our results on truncation for equation (3) in order to obtain, instead of the truncation for the KdV hierarchy, the truncation for the non-isospectral hierarchy $U_{t_{2n+1}} = \mathcal{R}^n U_y$. Further, if instead of setting $g = 0$ we iterate also on g, we obtain results for the hierarchy (3.12) of [3].

Example two. It is also straightforward to perform a generalized truncation for the modified version of (3) under the Miura map $U = V_x - V^2$ [18], i.e.,

$$V_t = \overline{\mathcal{R}}V_\tau, \qquad \overline{\mathcal{R}} = \partial_x^2 - 4V^2 - 4V_x\partial_x^{-1}V, \tag{35}$$

in order to obtain its Darboux transformation, Lax pair and non-isospectral condition (6). In this way, proceeding as described above, we obtain the truncation for the entire modified KdV (mKdV) hierarchy. We note in particular that since (under the Miura map)

we obtain the same equations between S, C^t and C^τ (15) for (35) as for (3), and since the singular manifold equations for the KdV equation and the mKdV equation are the same, then we obtain the result that the singular manifold equation for the t_{2n+1} flow of the mKdV hierarchy $V_{t_{2n+1}} = \overline{\mathcal{R}}^n V_x$ is also as in (32).

Example three. As in our first example above, iterating our truncation for (35) also gives the truncation for the non-isospectral hierarchy $V_{t_{2n+1}} = \overline{\mathcal{R}}^n V_y$ (when we take as starting point for our iteration the truncation of $V_{t_3} = \overline{\mathcal{R}} V_y$, i.e., (35) itself).

6. Conclusions

We have introduced a new method of obtaining truncation results for hierarchies of integrable PDEs, based on the use of equations associated to non-isospectral scattering problems. As an example we have recovered, in a very straightforward fashion, results first obtained by Weiss for the KdV hierarchy. Extensions of these results to related hierarchies have also been considered. The approach given here is easily extended to generalizations of the truncation process, since it depends only on being able to obtain the generalized form of the truncation for equations of the form $U_t = RU_\tau$, where R is the recursion operator of the hierarchy under study.

Acknowledgements

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