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Singular Manifold Method for an Equation in $2+1$ Dimensions

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Abstract

The Singular Manifold Method is presented as an excellent tool to study a $2+1$ dimensional equation in despite of the fact that the same method presents several problems when applied to $1+1$ reductions of the same equation. Nevertheless these problems are solved when the number of dimensions of the equation is increased.

1 Introduction

There are many different approaches to the study of nonlinear partial differential equations. Equations in $1+1$ dimensions are considered as the easiest part of the field. The different methods start usually with equations in $1+1$ dimensions and then, when the method has succeeded, the generalization to $2+1$ dimensions can be considered [17].

Among the different methods to study a given PDE, the Singular Manifold Method (SMM) [21] based in the Painlevé property [20] has been proved to be very effective. As it is well known, the Painlevé test is an algorithmic procedure that allows us to determine if the solutions of a PDE are singlevalued in the initial conditions. Essentially, for a PDE in $z_1 ... z_n$ variables, the Painlevé test requires that all the solutions of the PDE could be locally written as:

$$u(z_1, ...z_n) = \sum_{j=0}^{\infty} u_j(z_1, ...z_n) \left[\phi(z_1, ...z_n)\right]^{j-a}, \tag{1.1}$$

where a is an integer positive number and $\phi(z_1, ...z_n)$ a totally arbitrary differentiable function. Furthermore, once the Painlevé test has been checked for a given PDE, the SMM allows us to derive Bäcklund transformations, Lax pair, Darboux transformations and tau-functions for the PDE. Nevertheless we must remember that there are some problems related with Painlevé property, Painlevé test and SMM. We list some of them:

- One of the main criticisms to the Painlevé property is the fact that it is noninvariant under changes of dependent and/or independent variables. Many times it is not easy to identify the change of variables that allow us to write a PDE in a form such that the Painlevé test can be applied with success. Hodograph transformations can be sometimes used to this purpose [5]. In this sense we must say that precisely this ubiquity of a PDE, that can appear in many different forms depending of the variables that we have choosed, can be solved by means of the SMM. Actually, when we have been able to write a PDE in a form in which the SMM has been successful, this method provides us the **singular manifold equations** that can be considered as the **canonical form** of a PDE. We can conjecture that, if two PDE's have the same singular manifold equations, then there exists a transformation that relates the two equations since they are essentially the same [10].

- The usual SMM could be too restrictive when is applied to PDEs with several Painlevé branches. Modifications of SMM that includes the different branches simultaneously can be found in the following references: [7], [9], [10] and [12]. Once more the solution of this problem includes a bonus: If an equation has two Painlevé branches, the modification of the SMM provides us not only the right answer but the Miura transformations that relate our initial PDE to another PDE with just one Painlevé branch ([9], [10]).

- The SMM requires the truncation of the Painlevé series at the constant level $j = a$ and the annulation of all the coefficients in the different powers of ϕ. Sometimes this condition is very restrictive and needs to be modified [8]. Specially for some equations in $1+1$ dimensions the SMM imposes so many restrictions that there is not freedom enough to get nontrivial solutions and/or to introduce a spectral parameter [7].

This paper concerns specially with the last of the problems listed above. We recall the ARS (Ablowitz, Ramani, Segur) conjecture, [1], according to which a PDE has the Painlevé property if all of its reductions have such a property. Our main aim in this paper is just the opposite. We show that a PDE in $2 + 1$ can be much easier analyzed through the SMM than its reductions to $1 + 1$. In fact we present an equation in $2 + 1$ in which the SMM works very well. Nevertheless the SMM, when applied directly to its three simplest reductions to $1 + 1$ dimensions, presents one or several of the problems listed above. One can conclude that it is necessary to increase the number of dimensions in order to have sufficient freedom for the SMM not be too restrictive.

The plan of the paper is the following:

- In section 2 an equation in $2 + 1$ dimensions is proposed and it is proved that passes the Painlevé test.

- In section 3 a complete analysis of the $2 + 1$ equation is made by means of the SMM that allows us to obtain the Lax pair and Darboux transformations for the equation.

- Different reductions are presented in section 4. Their Lax pairs are also derived by reduction of the $2 + 1$ dimensional Lax pair.

- Conclusions are presented in section 5.

2 An equation in $2+1$ dimensions

The equation under scrutiny is the following one for a field h depending on $2+1$ variables x, y and z:

$$\left[h_{xxz} - \frac{3}{4}\left(\frac{h_{xz}^2}{h_z}\right) + 3h_x h_z\right]_x = h_{yz}. \tag{2.1}$$

Alternatively we can introduce a new dependent field $p(x,y,z)$ in order to write (2.1) as the following system:

$$h_z + p^2 = 0, \tag{2.2}$$

$$-p_y + p_{xxx} + \frac{3}{2}\, p\, h_{xx} + 3\, p_x h_x = 0. \tag{2.3}$$

We obtained (2.1) by searching $2+1$ integrable generalizations of peakon equations. Actually, (2.1) generalizes the Ermakov-Pinney equation [13] that, as has been proved in [15], is related by means of a reciprocal transformation to an equation with peakon solutions: the Degasperis-Procesi equation. Furthermore, (2.1) can be considered as a modified version of the generalized Hirota-Satsuma equation presented in [2] and [8] as a model for an incompressible fluid.

2.1 Painlevé test

To check if (2.1) passes the Painlevé test, the solutions should be written locally as [20]:

$$h = \sum_{j=0}^{\infty} h_j(x,y,z)\, [\phi(x,y,z)]^{j-a}. \tag{2.4}$$

Substitution of (2.4) into (2.1) gives us a polynomial in ϕ (we have used MAPLE to handle the calculation) of the type:

$$\sum_{j=0}^{\infty} C_j\, [\phi(x,y,z)]^{3j-3a-6} = 0 \tag{2.5}$$

The leading term ($j=0$) gives us:

$$a = 1, \tag{2.6}$$

$$h_0 = \phi_x. \tag{2.7}$$

The coefficient in h_j is:

$$\phi_x^5 \phi_z^3 \phi^{j-9} (j-1)(j-3)(j-4)(j+1) \tag{2.8}$$

which means that the equation has resonances in $j = 1, 3, 4$. It is not difficult to check that C_1, C_3 and C_4 are identically 0 for any value of h_1, h_3 and h_4. Consequently h admits a local expansion (2.4) in terms of four arbitrary functions ϕ, h_1, h_3 and h_4 which means that **(2.1) passes the Painlevé test** [20].

2.2 Reductions

There are three obvious reductions of (2.1).

- 1) $\frac{\partial h}{\partial y} = 0$

  Actually it is equivalent to the reduction $\frac{\partial \hat{h}}{\partial y} = \frac{\partial \hat{h}}{\partial x}$ if we redefine h as $h = \hat{h} + \frac{x}{3}$. With this reduction the equation (2.1) is

  $$\left[h_{xxz} - \frac{3}{4}\left(\frac{h_{xz}^2}{h_z}\right) + 3h_x h_z\right]_x = 0. \tag{2.9}$$

  or

  $$h_z = -p^2,$$
  $$0 = \left(2pp_{xx} - p_x^2 + 3p^2 h_x\right)_x. \tag{2.10}$$

  Integration of (2.10) gives us:

  $$pp_{xx} - \frac{p_x^2}{2} + 2Vp^2 + F(z) = 0, \qquad V_z = -\frac{3}{4}(p^2)_x$$

  that is the Ermakov-Pinney equation [13]. As it has been proved in [15] (see equation (2.15) of this reference), this equation arises through a reciprocal transformation from the Degasperis-Procesi equation [6] that is an equation with peakon solutions [11] similar to the celebrated Camassa-Holm equation [4]. As we see in the next section, the SMM is not effective when applied directly to (2.9). The number of conditions that the SMM requires is so large that only a few trivial solutions can be identified by this procedure.

- 2) $\frac{\partial h}{\partial z} = \frac{\partial h}{\partial x}$ This reduction yields trivially to the modified Korteveg de Vries equation:

  $$p_y - p_{xxx} + 6p^2 p_x = 0. \tag{2.11}$$

  The problem, in this case, is that (2.11) has two Painlevé branches (see [12]). Nevertheless, as it has been proved in [7], the SMM can be implemented by including both branches simultaneously. This generalization of the SMM has been applied successfully to many equations with two Painlevé branches [9], [10].

- 3) $\frac{\partial h}{\partial z} = \frac{\partial h}{\partial y}$ This reduction yields the 1 + 1 equation:

  $$\left[h_{xxz} - \frac{3}{4}\left(\frac{h_{xz}^2}{h_z}\right) + 3h_x h_z\right]_x = h_{zz}. \tag{2.12}$$

  or

  $$h_z = -p^2,$$
  $$-p_z + p_{xxx} + \frac{3}{2}ph_{xx} + 3p_x h_x = 0. \tag{2.13}$$

  The problem with the SMM for this equation is exactly the same that in the case of the reduction 1). The method happens to be too restrictive and it does not provide a spectral parameter.

We see in the next sections how to solve the problems by going to 2 + 1 dimensions.

3 Singular Manifold Method

In order to make the calculations much easier, it is convenient to write (2.1) in nonlocal form as the system

$$h_y = n_x, \tag{3.1}$$

$$h_{xxz}h_z - \frac{3}{4}h_{xz}^2 + 3h_x h_z^2 - h_z n_z = 0. \tag{3.2}$$

3.1 Truncated expansion

The SSM [21] requires the truncation of the Painlevé series (2.4) at the constant level $j = a$. It implies according to (2.6)-(2.7) that solutions $h^{(1)}, n^{(1)}$ of (3.2) can be written as:

$$\begin{aligned} h^{(1)} &= h + \frac{\phi_x}{\phi}, \\ n^{(1)} &= n + \frac{\phi_y}{\phi}. \end{aligned} \tag{3.3}$$

The SMM implies that h and n are also seminal solutions of the system (3.1)-(3.2). ϕ is now the so-called **singular manifold** associated to the (h, n) solution.

Substitution of (3.3) in (3.2) yields a polynomial in negative powers of ϕ of the form:

$$\sum_{k=0}^{4} E_k \left(\frac{\phi_x}{\phi}\right)^k = 0. \tag{3.4}$$

Setting to zero all the coefficients E_k, we get the following results (see Appendix A):

3.2 Seminal solutions

The seminal field h can be written in terms of the singular manifold through the following expressions:

$$h_x = -\frac{V_x}{3} - \frac{V^2}{12} + \frac{Q}{3}, \tag{3.5}$$

$$h_z = -\frac{1}{4R}\left(R_x + RV\right)^2, \tag{3.6}$$

where it has been useful to define V, R and Q as:

$$\begin{aligned} V &= \frac{\phi_{xx}}{\phi_x}, \\ R &= \frac{\phi_z}{\phi_x}, \\ Q &= \frac{\phi_y}{\phi_x}. \end{aligned} \tag{3.7}$$

The compatibility between the above definitions implies:

$$V_z = (R_x + RV)_x \tag{3.8}$$
$$V_y = (Q_x + QV)_x\,. \tag{3.9}$$

It is also convenient to define the Schwartzian derivative:

$$S = V_x - \frac{V^2}{2} \tag{3.10}$$

3.3 Singular Manifold Equations

E_k=0 provides (see Appendix A) the following relation between S, Q and R.

$$Q_z = S_z - \frac{3}{2}R_x\left(S + \frac{R_{xx}}{R} - \frac{R_x^2}{2R^2}\right) \tag{3.11}$$

that together with (3.8) and (3.9) constitute the **singular manifold equations**, which means the equations that the singular manifold ϕ should satisfy in order to have a truncation of the Painlevé series.

3.4 Lax pair

The expressions (3.5)-(3.6) can be easily linearized by introducing a new function ψ defined as:

$$\phi_x = \psi^2 \tag{3.12}$$

and consequently according to (3.7)-(3.9), we have:

$$V = 2\frac{\psi_x}{\psi}, \tag{3.13}$$
$$R_x + RV = 2\frac{\psi_z}{\psi}, \tag{3.14}$$
$$Q_x + QV = 2\frac{\psi_y}{\psi}, \tag{3.15}$$

By substituting (3.13) and (3.14) into (3.5)-(3.6), we get:

$$Q = 3h_x + 2\frac{\psi_{xx}}{\psi} - \frac{\psi_x^2}{\psi^2}, \tag{3.16}$$
$$R = -\frac{\psi_z^2}{h_z\psi^2}. \tag{3.17}$$

Substitution of (3.13), (3.16) and (3.17) into (3.14)-(3.15) provides us the Lax pair:

$$-\psi_y + \psi_{xxx} + 3h_x\psi_x + \frac{3}{2}h_{xx}\psi = 0, \tag{3.18}$$
$$\psi_{xz} - \frac{h_{xz}}{2h_z}\psi_z + h_z\psi = 0, \tag{3.19}$$

where the eigenfunction ψ is related to the singular manifold through (3.12).

If we consider the Lax pair as the compatibility condition between the operators

$$T_1 = \partial_x\partial_z - \frac{h_{xz}}{2h_z}\partial_z + h_z$$
$$T_2 = \partial_y - \partial_x^3 - 3h_x\partial_x - \frac{3}{2}h_{xx}$$

and compare with the spectral problem studied in [3] (see equations (2.1) and (2.2) of this reference), it is easy to see that T_1 belongs to the class discussed there, but T_2 does not because in our case it has ∂_x^3 instead of ∂_x^2 [17].

3.5 Darboux Transformations

Let ψ_1 and ψ_2 be two different eigenfunctions for h, which means that they satisfy (3.18)-(3.19).

$$\begin{aligned} &-\psi_{1,y} + \psi_{1,xxx} + 3h_x\psi_{1,x} + \frac{3}{2}h_{xx}\psi_1 = 0, \\ &\psi_{1,xz} - \frac{h_{xz}}{h_z}\psi_{1,z} + h_z\psi_1 = 0. \end{aligned} \tag{3.20}$$

$$\begin{aligned} &-\psi_{2,y} + \psi_{2,xxx} + 3h_x\psi_{2,x} + \frac{3}{2}h_{xx}\psi_2 = 0, \\ &\psi_{2,xz} - \frac{h_{xz}}{h_z}\psi_{2,z} + h_z\psi_2 = 0. \end{aligned} \tag{3.21}$$

Therefore there must be two singular manifolds for h defined as:

$$\phi_{1,x} = \psi_1^2, \qquad \phi_{2,x} = \psi_2^2. \tag{3.22}$$

According to (3.3) we can obtain a new solution $(h^{(1)}, n^{(1)})$ through the truncated expansion

$$\begin{aligned} h^{(1)} &= h + \frac{\phi_{1,x}}{\phi_1}, \\ n^{(1)} &= n + \frac{\phi_{1,y}}{\phi_1}. \end{aligned} \tag{3.23}$$

Since $(h^{(1)}, n^{(1)})$ is also solution of (3.1)-(3.2), its Lax pair is:

$$\begin{aligned} &-\psi_y^{(1)} + \psi_{xxx}^{(1)} + 3h_x^{(1)}\psi_x^{(1)} + \frac{3}{2}h_{xx}^{(1)}\psi^{(1)} = 0, \\ &\psi_{xz}^{(1)} - \frac{h_{xz}^{(1)}}{h_z^{(1)}}\psi_z^{(1)} + h_z^{(1)}\psi^{(1)} = 0, \end{aligned} \tag{3.24}$$

where $\psi^{(1)}$ is an eigenfuntion for $h^{(1)}$. So we can define a singular manifold $\phi^{(1)}$ for $h^{(1)}$ through the following expression

$$\phi_x^{(1)} = \left(\psi^{(1)}\right)^2. \tag{3.25}$$

The idea is to consider (3.1), (3.2), (3.24) and (3.25) as a nonlinear system of PDE's in $h^{(1)}$, $n^{(1)}$, $\psi^{(1)}$ and $\phi^{(1)}$. Therefore the truncated expansion (3.23) for $h^{(1)}$ and $n^{(1)}$ can be extended to $\psi^{(1)}$ and $\phi^{(1)}$ as:

$$\begin{aligned} \psi^{(1)} &= \psi_2 + \frac{\Lambda}{\phi_1}, \\ \phi^{(1)} &= \phi_2 + \frac{\Delta}{\phi_1}. \end{aligned} \tag{3.26}$$

Substitution of (3.23) and (3.26) in (3.24) and (3.25) provides us polynomials in negative powers of ϕ_1. The result (see Appendix B) is:

$$\begin{aligned} \Lambda &= -\psi_1 \Omega, \\ \Delta &= -\Omega^2, \end{aligned} \tag{3.27}$$

where Ω satisfies :

$$d\Omega = \psi_1 \psi_2 \, dx + (\psi_1 \psi_{2,xx} + \psi_2 \psi_{1,xx} - \psi_{1,x}\psi_{2,x} + 3h_x \psi_1 \psi_2) \, dy - \frac{\psi_{1,z}\psi_{2,z}}{h_z} \, dz \tag{3.28}$$

Therefore (3.23) and (3.26) are binary Darboux transformations in the sense that they allow us to construct the iterated Lax pair (3.24) through the solutions of two seminal Lax pairs (3.20) and (3.21). Nevertheless we should remark that these transformations are not the usual binary Darboux transformations that appear, for instance, in references [18] and [19], because in (3.26) not only the eigenfunctions ψ_1 and ψ_2 appear but also the singular manifold $\phi_1 = \int \psi_1^2 \, dx$. Transformations like (3.23)-(3.26) have been denominated Bäcklund-gauge transformations in reference [16].

3.6 Iterated solutions

As we have shown above, $\phi^{(1)}$ is a singular manifold for $h^{(1)}$. Therefore it can be used to iterate (3.23) in the following form:

$$\begin{aligned} h^{(2)} &= h^{(1)} + \frac{\phi_x^{(1)}}{\phi^{(1)}}, \\ n^{(2)} &= n^{(1)} + \frac{\phi_y^{(1)}}{\phi^{(1)}}. \end{aligned} \tag{3.29}$$

This second iteration (3.29) can be combined with the first one (3.23), and with (3.26) and (3.27) to give:

$$\begin{aligned} h^{(2)} &= h + \frac{\tau_x}{\tau}, \\ n^{(2)} &= n + \frac{\tau_y}{\tau}, \end{aligned} \tag{3.30}$$

where

$$\tau = \phi^{(1)} \phi_1 = \phi_2 \phi_1 - \Omega^2. \tag{3.31}$$

4 Reductions

As we said in subsection 2.2, there are several interesting reductions of (2.1).

4.1 $\frac{\partial h}{\partial y} = 0$

The equation can be written as:

$$\left[h_{xxz} - \frac{3}{4}\left(\frac{h_{xz}^2}{h_z}\right) + 3h_x h_z\right]_x = 0 \tag{4.1}$$

$$\begin{aligned} &or: h_z = -p^2, \\ &\left(2pp_{xx} - p_x^2 + 3p^2 h_x\right)_x = 0, \end{aligned} \tag{4.2}$$

that, after integration with respect to x, is the Ermakov-Pinney equation $pp_{xx} - \frac{p_x^2}{2} + 2Vp^2 + F(z) = 0$ with $V_z = -\frac{3}{4}(p^2)_x$ (see [13] and [15]). This reduction implies $Q = 0$. To get a right Lax pair with a spectral parameter it is necessary to go to the $2+1$ Lax pair (3.18)-(3.19) and make the gauge-reduction $\psi(x,y,z) = e^{\lambda y}\hat{\psi}(x,z)$. The reduction of (3.18)-(3.19) is obviously:

$$\hat{\psi}_{xxx} + 3h_x\hat{\psi}_x + \left(\frac{3}{2}h_{xx} - \lambda\right)\hat{\psi} = 0, \tag{4.3}$$

$$\hat{\psi}_{xz} - \frac{h_{xz}}{h_z}\hat{\psi}_x + h_z\hat{\psi} = 0. \tag{4.4}$$

The compatibility condition between (4.3) and (4.4) gives us the **third-order spectral problem**

$$\begin{aligned} &\hat{\psi}_{xxx} + 3h_x\hat{\psi}_x + \left(\frac{3}{2}h_{xx} - \lambda\right)\hat{\psi} = 0, \\ &\lambda\hat{\psi}_z = -h_z\psi_{xx} + \frac{h_{xz}}{2}\hat{\psi}_x + \left(-\frac{h_{xxz}}{2} + \frac{h_{xz}^2}{4h_z} - 3h_x h_z\right)\hat{\psi} = 0, \end{aligned} \tag{4.5}$$

that is the Lax pair of reference [14].

4.2 $\frac{\partial h}{\partial z} = \frac{\partial h}{\partial x}$

In this case the reduction of (2.2) is:

$$h_x = -p^2. \tag{4.6}$$

Therefore (2.3) is the modified Korteveg-de Vries equation:

$$-p_y + p_{xxx} - 6p^2 p_x = 0. \tag{4.7}$$

Before doing the reduction in the Lax pair, we need to introduce the spectral parameter through the gauge $\psi = e^{\lambda z}\hat{\psi}$ and then make the reduction $\frac{\partial h}{\partial z} = \frac{\partial h}{\partial x}$, $h_x = -p^2$ in (3.18)-(3.19). The result is the **second-order spectral problem**:

$$\begin{aligned} &\hat{\psi}_{xx} + \left(\lambda - \frac{p_x}{p}\right)\hat{\psi}_x + \left(-p^2 - \lambda\frac{p_x}{p}\right)\hat{\psi} = 0, \\ &\hat{\psi}_y + \left(2p^2 + \lambda\frac{p_x}{p} - \frac{p_{xx}}{p} - \lambda^2\right)\hat{\psi}_x + \left(\lambda\frac{p_{xx}}{p} + \lambda^2\frac{p_x}{p} + \lambda p^2\right)\hat{\psi} = 0. \end{aligned} \tag{4.8}$$

4.3 $\frac{\partial h}{\partial z} = \frac{\partial h}{\partial y}$

The reduction gives us the 1 + 1 PDE

$$\left[h_{xxz} - \frac{3}{4}\left(\frac{h_{xz}^2}{h_z}\right) + 3h_x h_z\right]_x = h_{zz}. \tag{4.9}$$

To have the right reduction of the Lax pair, we need to make the gauge transformation $\psi = e^{\lambda y}\hat{\psi}$ and the reduction $\frac{\partial h}{\partial z} = \frac{\partial h}{\partial y}$. In this case (3.18)-(3.19) reduce to:

$$-\hat{\psi}_z + \hat{\psi}_{xxx} + 3h_x\hat{\psi}_x + \left(\frac{3}{2}h_{xx} - \lambda\right)\hat{\psi} = 0, \tag{4.10}$$

$$\hat{\psi}_{xz} - \frac{h_{xz}}{h_z}\hat{\psi}_x + h_z\hat{\psi} = 0. \tag{4.11}$$

Solving (4.10) for $\hat{\psi}_z$ and substituting this into (4.11), we get the **fourth-order spectral problem**:

$$\begin{aligned} &\hat{\psi}_{xxxx} - \frac{h_{xz}}{2h_z}\hat{\psi}_{xxx} + 3h_x\hat{\psi}_{xx} + \\ &\qquad + \left(\frac{9h_{xx}}{2} - \frac{3h_x h_{xz}}{2h_z} - \lambda\right)\hat{\psi}_x + \left(h_z + \frac{3h_{xxx}}{2} + \frac{\lambda h_{xz}}{2h_z} - \frac{3h_{xz}h_{xx}}{4h_z}\right)\hat{\psi} = 0, \\ &-\hat{\psi}_z + \hat{\psi}_{xxx} + 3h_x\hat{\psi}_x + \left(\frac{3}{2}h_{xx} - \lambda\right)\hat{\psi} = 0. \end{aligned} \tag{4.12}$$

5 Conclusions

- A new equation (2.1) in 2 + 1 dimensions is presented and the Painlevé test is successfully applied.
- It should be noted that, although the SMM presents several problems when it is applied directly to different 1+1 dimensional reductions of (2.1), it has been proved to be an excellent tool to analyze the 2+1 dimensional equation (2.1). It works very well and allow us to obtain the Lax pair, Darboux transformations and an iterative method to obtain solutions of (2.1).
- The Lax pairs for the different 1+1 dimensional reductions can be derived from the 2 + 1 dimensional Lax pair.

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Appendix A

We used MAPLE to compute the polynomial that results from the substitution of (3.3) into (3.2): The result is a polynomial of the form (3.4) the first coefficient of which is:

$$E_4 = \frac{R^2}{4}\left(6h_x + 3v_x - S - 2Q\right) \tag{5.1}$$

that can be solved as:

$$h_x = -\frac{V_x}{2} + \frac{Q}{3} + \frac{S}{6}. \tag{5.2}$$

At this point it is useful to remember that the Painlevé Property is invariant under homographic transformations [21]. Therefore it is convenient to write everything in terms of the homographic invariants S, R and Q. It can be done if we introduce the change:

$$\begin{aligned} h &= \alpha - \frac{V}{2}, \\ n &= \beta - \frac{Q_x + QV}{2}. \end{aligned} \tag{5.3}$$

From (5.2) and (5.3) we have that α_x is the homographic invariant

$$\alpha_x = \frac{Q}{3} + \frac{S}{6} \tag{5.4}$$

Substitution of (5.2) and (5.3) into (3.4) gives us:

$$E_3 = 0$$

$$E_2 = \frac{1}{6}\left(-18\alpha_z^2 + 6R\alpha_z\left(S - Q\right) + 6R\beta_z + 2RS_{xz} - 5RQ_{xz}\right) \tag{5.5}$$

$$E_1 = -E_2 V - \frac{1}{6}\left(6\alpha_z(Q_z - S_z + SR_x - QR_x) - R_x(2S_{xz} - 5Q_{xz}) + 6R_x\beta_z\right). \tag{5.6}$$

We can solve (5.5) for β_z and by substituting it into (5.6) we have:

$$\alpha_z = \frac{R}{3R_x}\left(S_z - Q_z\right). \tag{5.7}$$

The compatibility condition $\alpha_{xz} = \alpha_{zx}$ between (5.4) and (5.7) provides us:

$$\left(S_z - Q_z\right)^2\left(-4R^2\left(S_z - Q_z\right) + R_x\left(-3R_x^2 + 6RR_{xx} + 6SR^2\right)\right) = 0. \tag{5.8}$$

The solution $S_z = Q_z$ of (5.8) it is not useful because in such a case (5.7) implies $\alpha_z = 0$. Therefore the solution of (5.8) should be:

$$Q_z = S_z + \frac{3}{2}R_x\left(S + \frac{R_{xx}}{R} - \frac{R_x^2}{2R^2}\right), \tag{5.9}$$

that is the singular manifold equation (3.11). Substitution of (5.9) into (5.7) yields:

$$\alpha_z = \frac{R_{xx}}{2} + \frac{RS}{2} - \frac{R_x^2}{4R}. \tag{5.10}$$

Substitution of (5.4) and (5.10) into (5.3) gives finally us (3.5)-(3.6).

Appendix B

• Substitution of (3.26) into the second of equations (3.24) gives us the following polynomial in ϕ_1

$$\left(\phi_{2,x} - \psi_2^2\right) + \frac{1}{\phi_1}\left(\Delta_x - 2\psi_2\Lambda\right) - \frac{1}{\phi_1^2}\left(\Delta\phi_{1,x} + \Lambda^2\right) = 0. \tag{5.11}$$

By using (3.22) we get:

$$\frac{1}{\phi_1}\left(\Delta_x - 2\psi_2\Lambda\right) - \frac{1}{\phi_1^2}\left(\Delta\psi_1^2 + \Lambda^2\right) = 0. \tag{5.12}$$

Setting to zero both coefficients we have:

$$\begin{aligned} &\Delta = -\left(\frac{\Lambda}{\psi_1}\right)^2, \\ &\left(\frac{\Lambda}{\psi_1}\right)_x + \psi_1\psi_2 = 0, \end{aligned} \tag{5.13}$$

that can be easily written as:

$$\begin{aligned} &\Delta = -\Omega^2, \\ &\Omega_x = \psi_1\psi_2, \end{aligned} \tag{5.14}$$

by introducing

$$\Omega = -\frac{\Delta}{\psi_1}. \tag{5.15}$$

• Substitution of (3.26) into the second of the equations (3.24) gives us:

$$\left(\psi_{1,x} - \psi_1\frac{h_{xz}}{2h_z}\right)\left[\frac{1}{\phi_1}\left(\Omega_z h_z + \psi_{1,z}\psi_{2,z}\right) - \frac{1}{\phi_1^2}\left(\Omega h_z\phi_{1,z} + \Omega\psi_{1,z}^2 + \phi_{1,z}\psi_{2,z}\psi_1 - \Omega_z\psi_{1,z}\psi_1\right)\right], \tag{5.16}$$

where we have used (5.14) and $\psi_{i,xz} = -h_z\psi_i + \dfrac{h_{xz}}{h_z}\psi_{i,z}, \quad i = 1, 2.$

From (3.6), (3.7) and (3.17) we have:

$$\phi_{1,z} = -\frac{\psi_{1,z}^2}{h_z} \tag{5.17}$$

that can be substituted into (5.16) to give:

$$\left(\psi_{1,x} - \psi_1\frac{h_{xz}}{2h_z}\right)\left(\frac{1}{\phi_1} + \frac{\psi_1\psi_{1,z}}{h_z\phi_1^2}\right)\left(\Omega_z h_z + \psi_{1,z}\psi_{2,z}\right). \tag{5.18}$$

Therefore

$$\Omega_z = -\frac{\psi_{1,z}\psi_{2,z}}{h_z} \tag{5.19}$$

• Substitution of (3.26) in the first of the equations (3.25) gives us the following polynomial in ϕ_1

$$\frac{\psi_1\Omega}{\phi_1^2}\left(2\psi_1\psi_{1,xx}-\psi_{1,x}^2+3h_x\psi_1^2-\phi_{1,y}\right)-\frac{\psi_1}{\phi_1}\Omega\left(\psi_{1,y}-\psi_{1,xxx}-3h_x\psi_{1,x}-\frac{3}{2}h_{xx}\psi_1\right)$$
$$-\frac{\psi_1}{\phi_1}\left(\psi_1\psi_{2,xx}+\psi_2\psi_{1,xx}-\psi_{1,x}\psi_{2,x}+3h_x\psi_1-\Omega_y\right)=0. \tag{5.20}$$

From (3.5) and (3.7) we have:

$$\phi_{1,y}=\phi_{1,x}\left(3h_x+V_{1,x}+\frac{V_1^2}{4}\right) \tag{5.21}$$

and, if we use (3.12) and (3.13),

$$\phi_{1,y}=3h_x\psi_1^2+2\psi_1\psi_{1,xx}-\psi_{1,x}^2. \tag{5.22}$$

Substitution of (5.22) and (3.20) into (5.20) provides us:

$$-\frac{\psi_1}{\phi_1}\left(\psi_1\psi_{2,xx}+\psi_2\psi_{1,xx}-\psi_{1,x}\psi_{2,x}+3h_x\psi_1-\Omega_y\right)=0 \tag{5.23}$$

and therefore:

$$\Omega_y=\psi_1\psi_{2,xx}+\psi_2\psi_{1,xx}-\psi_{1,x}\psi_{2,x}+3h_x\psi_1. \tag{5.24}$$

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