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Hodograph transformations for a Camassa–Holm hierarchy in 2 + 1 dimensions

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Abstract

A generalization of the negative Camassa–Holm hierarchy to 2 + 1 dimensions is presented under the name CHH(2+1). Several hodograph transformations are applied in order to transform the hierarchy into a system of coupled CBS (Calogero–Bogoyavlenskii–Schiff) equations in 2 + 1 dimensions that pass the Painlevé test. A non-isospectral Lax pair for CHH(2+1) is obtained through the above-mentioned relationship with the CBS spectral problem.

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1. Introduction

The seminal papers in which the Camassa–Holm equation was described [4, 5] have led to much work related to equations with peakon solutions. In particular, in [12, 8, 15], the authors include the Camassa–Holm equation within a wider class of equations with peakons. The integrability of the Camassa–Holm equation, spectral problem, solutions, etc have been studied in many papers in the last 10 years (see, for instance, [4, 8, 15]).

The Painlevé test [19] is usually presented as a powerful instrument to check the integrability of an equation. Nevertheless, in [12] the limitations of the Painlevé test when applied to Camassa–Holm-like equations are discussed.

The Painlevé property provides not only the basis for the Painlevé test, but also for the singular manifold method [19]. When an equation passes the Painlevé test, the singular manifold method can be applied to algorithmically construct the Lax pair [9, 10] and many other properties of integrable systems such as Darboux transformations, τ-functions, etc. The main problem with Painlevé methods is that the Painlevé property is non-invariant under changes of independent and/or dependent variables. Often, finding the change of variables that writes an equation in a form that passes the Painlevé test, is a question of luck or ability.

From the point of view of the spectral problem, the Lax pair for a partial differential equation is usually found by inspection. Most frequently, a spectral problem is proposed and

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then the equations that satisfy this spectral problem are derived [18, 1, 6]. In contrast, the singular manifold method has the attractive property that it allows us to start from a given equation (that passes the Painlevé test) and derive its Lax pair in a very precise way. Our conjecture is that if an equation is integrable, there must be a transformation that will let us transform the equation into a new one in which the Painlevé test is successful and the singular manifold method can be applied to derive the Lax pair.

In [8, 14], hodograph transformations were proposed as useful instruments to transform peakon equations into equations that pass the Painlevé test. On the basis of this idea, in section 2, we attempt to study the integrability of an n-component Camassa–Holm hierarchy in 2+1 dimensions (which we will call CHH(2+1)) by means of several hodograph transformations that map this hierarchy in a system of n coupled CBS (Calogero–Bogoyavlenskii–Schiff [3, 7]) equations in three independent variables that are different for each CBS component. This result generalizes those obtained in [14], where reciprocal transformations between the first component of the CHH(2+1) and CBS are studied.

The CBS equation in three dimensions has been proved to pass the Painlevé test [10]. In the same reference, the singular manifold method was used to construct the Lax pair, which in fact is a non-isospectral one [6, 10]. This knowledge of the spectral problem associated with the CBS equation allows us to devote section 3 to reversing the hodograph transformations and rewrite the spectral problem in the original variables. Thus, a non-isospectral Lax pair for the CHH(2+1) hierarchy is obtained. The coincidences and differences between these results and other spectral problems are discussed at the end of this section.

The conclusions are presented in section 4.

2. Camassa–Holm hierarchy in 2+1 dimensions

- As is well known [13], the negative Camassa–Holm hierarchy for a field $u(x, t)$ can be written as

$$u_t = R^{-n} u_x, \qquad R = J_0 J_1^{-1} \quad n \geqslant 1, \tag{2.1}$$

where n is an integer number that is the order of the hierarchy and J_0, J_1 are the following operators:

$$J_0 = (\partial^3 - \partial), \qquad J_1 = (u\partial + \partial u), \qquad \partial = \frac{\partial}{\partial x}. \tag{2.2}$$

For our purpose it is convenient to introduce n functions $v_1(x, t), \dots, v_n(x, t)$ defined as

$$\begin{aligned} v_1 &= J_0^{-1} u_x \quad \Longrightarrow \quad J_0 v_1 = u_x \\ v_j &= J_0^{-1} J_1 v_{j-1} \quad \Longrightarrow \quad J_0 v_j = J_1 v_{j-1}, \qquad j = 2, \dots, n. \end{aligned} \tag{2.3}$$

Equation (2.1) can now be written simply as

$$u_t = J_1 v_n, \tag{2.4}$$

and hence the negative Camassa–Holm hierarchy can be considered as the $n + 1$ equations (2.3) and (2.4) in $n + 1$ fields $u, v_1, \dots, v_n$.
Obviously, for $n = 1$, the system (2.3)–(2.4) reduces to

$$u_t = 2u(v_1)_x + u_x v_1 \qquad u = (v_1)_{xx} - v_1, \tag{2.5}$$

which is the celebrated Camassa–Holm equation [5].
- The positive Camassa–Holm hierarchy [13] would be obtained through

$$u_t = R^n(0), \qquad n \geqslant 1, \tag{2.6}$$

whose $n = 1$ component is

$$u_t = J_0 J_1^{-1}(0);$$

or equivalently

$$u_t = J_0 v_1 \qquad J_1 v_1 = 0 \quad \Longrightarrow \quad v_1 = u^{-1/2},$$

which is the Dym equation [16] with an extra term $(v_1)_x$ [6, 1].

2.1. Generalization to three dimensions

A simple generalization of (2.3)–(2.4) to three dimensions is as follows:

$$\begin{aligned} U_y &= J_0 V_1 \\ J_0 V_j &= J_1 V_{j-1}, \qquad j = 2, \ldots, n, \\ U_t &= J_1 V_n, \end{aligned} \tag{2.7}$$

where $U = U(x, t, y)$, $V_j = V_j(x, t, y)$.

System (2.7) can also be written as

$$U_t = R^{-n} U_y. \tag{2.8}$$

The equivalent positive hierarchy should be

$$U_t = R^n U_y, \tag{2.9}$$

which can be trivially obtained from (2.8) by interchanging t and y. Consequently, in three dimensions (2.8) contains both the negative and positive hierarchies. One can be obtained from the other by interchanging the roles of t and y.

It is also necessary to point out that the first component of (2.7) can be written (by simply putting $V_1 = m_y$) as

$$(\partial_t - 2m_{xy} - m_y \partial_x)(m_{xxx} - m_x) = 0$$

that is a generalization to 2 + 1 dimensions of the Fokas–Fuchssteiner–Camassa–Holm equation [11] proposed in [6] and analysed in [14].

Reductions

- It is trivial to see that the negative Camassa–Holm hierarchy (2.1) would be obtained from (2.8) through the reduction $\frac{\partial}{\partial y} = \frac{\partial}{\partial x}$.
- If we reduce (2.8) by setting $\frac{\partial}{\partial t} = 0$ we obtain

  $$R^{-n} U_y = 0 \quad \Longrightarrow \quad U_y = R^n(0)$$

  which is the positive hierarchy (2.6) where t has been replaced by y.

Note that (2.8) is formally included in the Dym case of [6]. Nevertheless, the generalization of the Camassa–Holm hierarchy that the authors construct explicitly in that work is not (2.9) because it corresponds to $n = 1$ and U is a field with N components (see equation (2.16) of this reference). Only the first components of both hierarchies ($n = N = 1$) coincide.

Below we shall denote (2.7) by CHH(2+1) and prove through several hodograph transformations that it can be transformed into a system that passes the Painlevé test.

2.2. First hodograph transformation

If we set

$$U = P^2, \tag{2.10}$$

we can write system (2.7) as

$$P_y = (\beta_1)_x, \tag{2.11}$$

$$\frac{J_0 V_j}{2P} = (PV_{j-1})_x, \qquad j = 2, \ldots, n, \tag{2.12}$$

$$P_t = (PV_n)_x, \tag{2.13}$$

where we have defined

$$(\beta_1)_x = \frac{J_0 V_1}{2P}. \tag{2.14}$$

The conservative form of (2.11) and (2.13) allows us, according to [14, 8], to define the following hodograph transformation:

$$\mathrm{d}X = P\,\mathrm{d}x + PV_n\,\mathrm{d}t + \beta_1\,\mathrm{d}y, \qquad Z_1 = t, \qquad Y = y, \tag{2.15}$$

The partial derivatives are now

$$\frac{\partial}{\partial x} = P\frac{\partial}{\partial X}, \qquad \frac{\partial}{\partial t} = \frac{\partial}{\partial Z_1} + PV_n\frac{\partial}{\partial X}, \qquad \frac{\partial}{\partial y} = \frac{\partial}{\partial Y} + \beta_1\frac{\partial}{\partial X}. \tag{2.16}$$

The inverses of (2.15) and (2.16) are

$$\mathrm{d}x = \frac{\mathrm{d}X}{P} - V_n\,\mathrm{d}Z_1 - \frac{\beta_1}{P}\,\mathrm{d}Y, \qquad t = Z_1, \qquad y = Y, \tag{2.17}$$

$$\frac{\partial}{\partial X} = \frac{1}{P}\frac{\partial}{\partial x}, \qquad \frac{\partial}{\partial Z_1} = \frac{\partial}{\partial t} - V_n\frac{\partial}{\partial x}, \qquad \frac{\partial}{\partial Y} = \frac{\partial}{\partial y} - \frac{\beta_1}{P}\frac{\partial}{\partial x}. \tag{2.18}$$

With this hodograph transformation, system (2.11)–(2.14) becomes

$$P_Y = P(\beta_1)_X - P_X\beta_1 \tag{2.19}$$

$$\frac{P_{Z_1}}{P^2} = (V_n)_X \tag{2.20}$$

$$\frac{1}{2P}(\{P[P(V_j)_X]_X\}_X - (V_j)_X) = (PV_{j-1})_X, \qquad j = 2, \ldots, n \tag{2.21}$$

$$\frac{1}{2P}(\{P[P(V_1)_X]_X\}_X - (V_1)_X) = (\beta_1)_X. \tag{2.22}$$

Nevertheless, (2.19)–(2.22) is not yet a system in which the Lax pair can be directly derived. A new set of transformations is needed in order to write (2.19)–(2.22) in a form in which the singular manifold method could be applied to derive the Lax pair.

2.3. Second hodograph transformation

- Let us take (2.21) for $j = n$:

  $$\frac{1}{2P}(\{P[P(V_n)_X]_X\}_X - (V_n)_X) = (PV_{n-1})_X,$$

  and by substituting (2.20), the result is

  $$\left(\frac{P_{XX}}{2P} + \frac{1 - P_X^2}{4P^2}\right)_{Z_1} = (PV_{n-1})_X. \tag{2.23}$$

  The form of equation (2.23) suggests that we should introduce a new function H, defined as

  $$H_X = \left(\frac{P_{XX}}{2P} + \frac{1 - P_X^2}{4P^2}\right), \tag{2.24}$$

which allows us to integrate (2.23) as

$$PV_{n-1} = H_{Z_1}. \tag{2.25}$$

- On the basis of (2.25), let us introduce $Z_2, \ldots, Z_{n-1}$ new independent variables, such that equation (2.25) can be extended through the following definition:

$$PV_{n-j} = H_{Z_j}, \qquad j = 2, \ldots, n-1 \quad \Longrightarrow \quad PV_j = H_{Z_{n-j}}, \qquad j = 1, \ldots, n-2. \tag{2.26}$$

  Note that (2.26) are hodograph transformations between each dependent variable V_j and the corresponding independent variable Z_{n-j}.
- Taking (2.21) for $n - j$,

$$\frac{1}{2P}(\{P[P(V_{n-j})_X]_X\}_X - (V_{n-j})_X) = (PV_{n-j-1})_X, \qquad j = 1, \ldots, n-2,$$

  and by using (2.26)

$$\frac{1}{2P}\left(\left\{P\left[P\left(\frac{H_{Z_j}}{P}\right)_X\right]_X\right\}_X - \left(\frac{H_{Z_j}}{P}\right)_X\right) = \left(H_{Z_{j+1}}\right)_X. \tag{2.27}$$

  We can use (2.24) to obtain

$$P_{XX} = 2PH_X + \frac{P_X^2 - 1}{2P}. \tag{2.28}$$

  By substituting (2.28) in (2.27), we have

$$H_{XXXZ_j} - 4H_{XZ_j}H_X - 2H_{XX}H_{Z_j} = 2H_{XZ_{j+1}}, \qquad j = 1, \ldots, n-2. \tag{2.29}$$

  Each of the equations of (2.29) is a CBS equation in the three variables X, Z_j and Z_{j+1}. This equation has been studied by different authors (see [7, 3, 6, 10, 14]). This equation can also be considered as a generalization to 2 + 1 dimensions of the AKNS (Ablowitz, Kaup, Neweel, Segur) equation. Its Lax pair can be found through the singular manifold method in [10] and it proves to be non-isospectral [6, 17, 10]. We shall use this result in the next section.

2.4. *Third hodograph transformation*

- By substituting $V_1 = \frac{H_{Z_{n-1}}}{P}$ in (2.22), we have

$$H_{XXXZ_{n-1}} - 4H_{XZ_{n-1}}H_X - 2H_{XX}H_{Z_{n-1}} = 2(\beta_1)_X. \tag{2.30}$$

  We now define a new variable Z_n such that

$$\beta_1 = H_{Z_n}, \tag{2.31}$$

  which is again a hodograph transformation between the dependent variable β_1 and the independent one Z_n. With this transformation, (2.30) looks exactly like (2.29) for $j = n$.

$$H_{XXXZ_{n-1}} - 4H_{XZ_{n-1}}H_X - 2H_{XX}H_{Z_{n-1}} = 2H_{XZ_n}. \tag{2.32}$$

  Thus, by combining (2.29) and (2.32), we have the following $n - 1$ CBS equations:

$$H_{XXXZ_j} - 4H_{XZ_j}H_X - 2H_{XX}H_{Z_j} = 2H_{XZ_{j+1}}, \qquad j = 1, \ldots, n-1. \tag{2.33}$$

- Substitution of (2.31) in (2.19) gives us

$$P_Y = PH_{XZ_n} - P_XH_{Z_n}, \tag{2.34}$$

  whose compatibility with (2.28) yields

$$H_{XXXZ_n} - 4H_{XZ_n}H_X - 2H_{XX}H_{Z_n} = 2H_{XY}, \tag{2.35}$$

  which is again a CBS equation in the variables X, Z_n and Y.

2.5. Summary of the transformations

We now summarize the above results.

Let us start with the CHH(2+1) system given by (2.11)–(2.14). This is a system of n fields $V_1, \ldots, V_n$ and three independent variables: x, t and y. We have made the following transformations:

(1)

$$\mathrm{d}X = P\mathrm{d}x + PV_n\,\mathrm{d}t + \beta_1\,\mathrm{d}y, \qquad Z_1 = t, \qquad Y = y. \tag{2.36}$$

(2)

$$P_{XX} = 2PH_X + \frac{P_X^2 - 1}{2P} \tag{2.37}$$

$$P_Y = PH_{XZ_n} - P_X H_{Z_n} \tag{2.38}$$

$$P_{Z_1} = P^2(V_n)_X. \tag{2.39}$$

(3)

$$H_{Z_{n-j}} = PV_j, \qquad j = 1, \ldots, n-1. \tag{2.40}$$

(4)

$$H_{Z_n} = \beta_1. \tag{2.41}$$

With these transformations, we obtain the following system:

$$H_{XXXZ_j} - 4H_{XZ_j}H_X - 2H_{XX}H_{Z_j} = 2H_{XZ_{j+1}}, \qquad j = 1, \ldots, n-1 \tag{2.42}$$

$$H_{XXXZ_n} - 4H_{XZ_n}H_X - 2H_{XX}H_{Z_n} = 2H_{XY}. \tag{2.43}$$

We have now n CBS equations for just one field H and $n+2$ independent variables: $X, Y, Z_1, \ldots, Z_n$. It is fairly trivial to check that equations such as (2.42) pass the Painlevé test [10]. Consequently, the above-described hodograph transformations map the CHH(2+1) to a new system in which the Painlevé techniques (singular manifold method) can be applied. For $n = 1$ this corresponds to the result obtained by Hone in [14].

Note that after the first reciprocal transformation, the system was (2.19)–(2.22) where obviously P, V_i and β_1 are considered as independent fields. The second and third hodograph transformations

$$PV_j = H_{Z_{n-j}}, \qquad j = 1, \ldots, n-1$$
$$\beta_1 = H_{Z_n}$$

imply that, for any of the n independent fields $V_1, \ldots, V_{n-1}$ and β_1, we define one of the n variables $Z_1, \ldots, Z_n$, which consequently are as independent as the $V_1, \ldots, V_{n-1}, \beta_1$ fields are. Furthermore, in the appendix we will use the results of [10] to construct solitonic solutions of (2.42) and (2.43) depending on the $n+2$ independent variables: $X, Y, Z_1, \ldots, Z_n$. The main benefit of the second and third hodograph transformations is that they allow us to write the equations in a form in which the Lax pair can be algorithmically derived through the techniques of the singular manifold method.

We should remark that the hodograph transformation (2.36) is not defined for peakons. Actually, as has been pointed in [8, 15], (2.10) breaks down when U is a Dirac delta function because the square root of a distribution is not defined.

3. Integrability and Lax pair for CHH(2+1)

In a recent paper by us [10], the singular manifold method [19] was applied to CBS to derive its Lax pair. By using these results, the Lax pair for (2.42) is

$$\psi_{XX} = \left(H_X + \frac{\lambda}{2}\right)\psi. \tag{3.1}$$

$$0 = E_j = -\psi_{Z_{j+1}} + \lambda\psi_{Z_j} - H_{Z_j}\psi_X + \frac{H_{XZ_j}}{2}\psi, \qquad j = 1, \ldots, n-1. \tag{3.2}$$

For (2.43), the spatial part is exactly the same, but the temporal part is

$$0 = E_n = -\psi_Y + \lambda\psi_{Z_n} - H_{Z_n}\psi_X + \frac{H_{XZ_n}}{2}\psi. \tag{3.3}$$

Furthermore, the compatibility condition between (3.1) and (3.2) implies that the spectral problem is non-isospectral because λ satisfies

$$\lambda_X = 0, \qquad \lambda_{Z_{j+1}} - \lambda\lambda_{Z_j} = 0. \tag{3.4}$$

Analogously, the compatibility condition between (3.1) and (3.3) yields

$$\lambda_X = 0, \qquad \lambda_Y - \lambda\lambda_{Z_n} = 0. \tag{3.5}$$

Note that (3.1) is independent of the index j. Nevertheless, (3.2) can be considered as a recursion relation for the derivatives of ψ with respect to each Z_j. This allows us to take the following combination:

$$\begin{aligned} 0 &= E_n\lambda^{-n} + \sum_{j=1}^{n-1} E_j\lambda^{-j} \\ &= -\lambda^{-n}\psi_Y + \lambda^{1-n}\psi_{Z_n} - \lambda^{-n}H_{Z_n}\psi_X + \lambda^{-n}\frac{H_{XZ_n}}{2}\psi \\ &\quad + \sum_{j=1}^{n-1}\left[-\lambda^{-j}\psi_{Z_{j+1}} + \lambda^{1-j}\psi_{Z_j} - \lambda^{-j}H_{Z_j}\psi_X + \lambda^{-j}\frac{H_{XZ_j}}{2}\psi\right]. \end{aligned} \tag{3.6}$$

It is easy to see that

$$\sum_{j=1}^{n-1}\left[-\lambda^{-j}\psi_{Z_{j+1}}\right] + \sum_{j=1}^{n-1}\left[\lambda^{-j+1}\psi_{Z_j}\right] = -\lambda^{1-n}\psi_{Z_n} + \psi_{Z_1}.$$

Therefore, we have

$$0 = \psi_{Z_1} - \lambda^{-n}\psi_Y + \sum_{j=1}^{n}\left[-\lambda^{-j}H_{Z_j}\psi_X + \lambda^{-j}\frac{H_{XZ_j}}{2}\psi\right]. \tag{3.7}$$

The combination of (3.4) and (3.5) gives us

$$\lambda_Y - \lambda^n\lambda_{Z_1} = 0. \tag{3.8}$$

3.1. Inverse transformation

We can now come back to the original fields U and V_j as well as to the original independent variables x, t and y. All we need is to perform the change [14]

$$\psi(X, Z_1, \ldots, Z_n, Y) = \sqrt{P}\phi(x, t, y). \tag{3.9}$$

And, according to (2.18), we have

$$\begin{aligned} &\psi_X = \sqrt{P}\left(\frac{\phi_x}{P} + \frac{P_X}{2P}\phi\right) && \psi_{XX} = \sqrt{P}\left(\frac{\phi_{xx}}{P^2} + \left[\frac{P_{XX}}{2P} - \frac{P_X^2}{4P^2}\right]\phi\right) \\ &\psi_{Z_1} = \sqrt{P}\left(\phi_t - V_n\phi_x + \frac{P_{Z_1}}{2P}\phi\right) && \psi_Y = \sqrt{P}\left(\phi_y - \frac{\beta_1}{P}\phi_x + \frac{P_Y}{2P}\phi\right). \end{aligned} \tag{3.10}$$

With these changes, (3.1) becomes

$$\frac{\phi_{xx}}{P^2} + \left[\frac{P_{XX}}{2P} - \frac{P_X^2}{4P^2}\right]\phi = \left(H_X + \frac{\lambda}{2}\right)\phi.$$

Or, by using (2.37) and (2.10)

$$\phi_{xx} = \left(\frac{1}{4} + \frac{\lambda}{2}U\right)\phi, \tag{3.11}$$

which is the spatial part of CHH(2+1). The temporal part can be obtained by using (3.10) in (3.7). The result is

$$\begin{aligned} 0 = &\left[\phi_t - V_n\phi_x + \frac{P_{Z_1}}{2P}\phi\right] - \lambda^{-n}\left[\phi_y - \frac{\beta_1}{P}\phi_x + \frac{P_Y}{2P}\phi\right] \\ &+ \sum_{j=1}^{n}\lambda^{-j}\left[-H_{Z_j}\left(\frac{\phi_x}{P} + \frac{P_X}{2P}\phi\right) + \frac{H_{XZ_j}}{2}\phi\right]. \end{aligned}$$

We now need to use (2.38) and (2.39) to obtain

$$\begin{aligned} 0 = &\left[\phi_t - V_n\phi_x + \frac{P(V_n)_X}{2}\phi\right] - \lambda^{-n}\left[\phi_y - \frac{\beta_1}{P}\phi_x + \frac{PH_{XZ_n} - P_X H_{Z_n}}{2P}\phi\right] \\ &+ \sum_{j=1}^{n-1}\lambda^{-j}\left[-H_{Z_j}\left(\frac{\phi_x}{P} + \frac{P_X}{2P}\phi\right) + \frac{H_{XZ_j}}{2}\phi\right] \\ &- \lambda^n H_{Z_n}\left(\frac{\phi_x}{P} + \frac{P_X}{2P}\phi\right) + \lambda^{-n}\frac{H_{XZ_n}}{2}\phi, \end{aligned}$$

which can be simplified to

$$\begin{aligned} 0 = &\phi_t - V_n\phi_x + \frac{P(V_n)_X}{2}\phi - \lambda^{-n}\left[\phi_y + \left(-\frac{\beta_1}{P} + \frac{H_{Z_n}}{P}\right)\phi_x\right] \\ &+ \sum_{j=1}^{n-1}\lambda^{-j}\left[-H_{Z_j}\left(\frac{\phi_x}{P} + \frac{P_X}{2P}\phi\right) + \frac{H_{XZ_j}}{2}\phi\right]. \end{aligned}$$

By using (2.40) and (2.41), we have:

$$\begin{aligned} 0 = &\phi_t - V_n\phi_x + \frac{P(V_n)_X}{2}\phi - \lambda^{-n}\phi_y \\ &+ \sum_{j=1}^{n-1}\lambda^{-j}\left[-V_{n-j}\left(\phi_x + \frac{P_X}{2}\phi\right) + \frac{P(V_{n-j})_X + P_X V_{n-j}}{2}\phi\right], \end{aligned}$$

and simplifying

$$\phi_t - \lambda^{-n}\phi_y - \left[V_n + \sum_{j=1}^{n-1}\lambda^{-j}V_{n-j}\right]\phi_x + \frac{1}{2}\left[V_n + \sum_{j=1}^{n-1}\lambda^{-j}V_{n-j}\right]_x \phi = 0. \tag{3.12}$$

Furthermore, by applying (2.18) to (3.8), we have the non-isospectral condition

$$\lambda_y - \lambda^n\lambda_t = 0. \tag{3.13}$$

In sum, the Lax pair for the hierarchy CHH(2+1) of equations in 2 + 1 variables (2.7) can be written as

$$\phi_{xx} - \frac{\lambda}{2}U\phi = \frac{1}{4}\phi \tag{3.14}$$

$$\phi_t = \lambda^{-n}\phi_y + A\phi_x - \frac{A_x}{2}\phi, \tag{3.15}$$

where

$$A = \sum_{j=0}^{n-1}[\lambda^{-j}V_{n-j}], \qquad \lambda_y - \lambda^n\lambda_t = 0. \tag{3.16}$$

We have proved in [10] the usefulness of the Lax pair (3.1)–(3.2) for solving CBS. Actually, in [10] we have used the singular manifold method to obtain Darboux transformations for this Lax pair. These Darboux transformations are the basis for the construction of an iterative and algorithmic procedure described in [10] that allows us to obtain a rich collection of nontrivial solutions. The inversion of the hodograph transformations (2.36)–(2.41) provides us the corresponding solutions for CHH(2+1) and its Lax pair. It will be the subject of future work.

Remarks. Spectral problems similar to (3.10) have been considered in several papers [18, 1, 6, 13, 14]. More precisely,

- this Lax pair is included in the scattering problem presented in equation (1.1) of [6] and it corresponds to the case that these authors call the Dym case. Nevertheless, CHH(2+1) is not included in the cases that the authors presented explicitly because the generalization of the Camassa–Holm hierarchy that they considered corresponds to $n = 1$ (interchanging t and y) and U expanded as a polynomial of degree $N - 1$ in λ. Only the $n = 1$ component of CHH(2+1) is equivalent to equation (2.21) of [6] (the $N = 1$ case). The Lax pair for the $n = 1$ component of the hierarchy appears also in [14].
- The Lax pair considered in [13] for the negative Camassa–Holm hierarchy (2.3)–(2.4) can be obtained through the reduction $\frac{\partial}{\partial y} = \frac{\partial}{\partial x}$. Equivalently, the Lax pair presented in the same reference [13] for the positive Camassa–Holm hierarchy arises from the reduction $\frac{\partial}{\partial t} = 0$. In our notation, these 1 + 1 Lax pairs are

$$\begin{aligned}
&\phi_{xx} = \left(\frac{\lambda}{2}u + \frac{1}{4}\right)\phi \\
&\phi_t = B\phi_x - \frac{B_x}{2}\phi \\
&B = \lambda^{-n} + \sum_{j=0}^{n-1}\left[\lambda^{-j}V_{n-j}\right], \qquad \text{for the negative hierarchy,}
\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}\phi_{xx} &= \left(\frac{\lambda}{2}u + \frac{1}{4}\right)\phi \\ \phi_y &= C\phi_x - \frac{C_x}{2}\phi \\ C &= -\sum_{j=0}^{n-1}[\lambda^{n-j}V_{n-j}], \qquad \text{for the positive hierarchy.}\end{aligned} \tag{3.18}$$

- Equation (3.18) corresponds to the $N = 1$ (interchanging t and y) case of [1] (which generalizes [18]). Equation (3.17) is not included in this reference because expansions in negative powers of λ were not considered there.

4. Conclusions

Here we have presented an extension of the n-component Camassa–Holm hierarchy to 2+1 dimensions whose $n = 1$ component is a generalization of the Fokas–Fuchsteiner–Camassa–Holm equation. Although the Painlevé test cannot be applied to this system, we have found a set of hodograph transformations that allows us to transform the original CHH(2+1) into n coupled CBS equations that pass the Painlevé test. This result generalizes [14] for an n-component hierarchy. The relationship between integrable systems and the Painlevé property is once again established.

CBS is known to have a non-isospectral Lax pair. This Lax pair was used in section 3 to invert the hodograph transformations in order to obtain a non-isospectral Lax pair for CHH(2+1). Note that the non-isospectral condition $\lambda_y = \lambda^n\lambda_t$ depends on the order n of the hierarchy.

The Lax pairs for the positive and negative 1 + 1 Camassa–Holm hierarchies can be obtained through the reductions $\frac{\partial}{\partial t} = 0$ and $\frac{\partial}{\partial y} = \frac{\partial}{\partial x}$, respectively.

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Appendix

There are a lot of solutions of the coupled CBS equations (2.42) and (2.43) that can be obtained by using the techniques of [10]. The simplest solution can be constructed through the eigenfunctions of (3.1)–(3.3) with $H = 0$ and λ constant. These eigenfunctions can be written as

$$\psi = \exp\left(kX + \omega Y + \omega\sum_{j=1}^{n}\lambda^{(j-n-1)}Z_j\right) \tag{A.1}$$

where ω is a totally arbitrary constant and

$$k^2 = \frac{\lambda}{2}$$

According to [10], it allows us to construct the following singular manifold:

$$\phi \sim 1 + \exp\left(2(kX + \omega Y + \omega\sum_{j=1}^{n}\lambda^{(j-n-1)}Z_j)\right) \tag{A.2}$$

which yields the following one-soliton solution:

$$H_{1\text{-soliton}} = -2\frac{\phi_x}{\phi}. \tag{A.3}$$

A two-soliton solution can be easily written by means of the same techniques as [10] (see expressions (3.24) and (3.25) of this reference). The result is

$$H_{2\text{-soliton}} = -2\frac{\tau_x}{\tau} \tag{A.4}$$

where

$$\tau = \phi_1\phi_2 - \Omega^2. \tag{A.5}$$

ϕ_1 and ϕ_2 are singular manifolds of the form (A.2) corresponding to two different spectral parameters λ_1 and λ_2 and two different values ω_1 and ω_2 of ω.

$$\Omega = \frac{\psi_1\psi_{2,x} - \psi_1\psi_{2,x}}{\lambda_2 - \lambda_1} \tag{A.6}$$

which implies

$$\tau \sim 1 + \psi_1^2 + \psi_2^2 + \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 \psi_1^2\psi_2^2 \tag{A.7}$$

where

$$\psi_1 = \exp\left(k_1 X + \omega_1 Y + \omega_1 \sum_{j=1}^{n} \lambda_1^{(j-n-1)} Z_j\right) \tag{A.8}$$

$$\psi_2 = \exp\left(k_2 X + \omega_2 Y + \omega \sum_{j=1}^{n} \lambda_2^{(j-n-1)} Z_j\right) \tag{A.9}$$

and

$$k_1^2 = \frac{\lambda_1}{2}, \qquad k_2^2 = \frac{\lambda_2}{2}.$$

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