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Intense laser interacting with a two-level atom: WKB expressions for dipole transitions and population inversion

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Abstract

In a previous paper, we have already considered the system composed by a two-level atom interacting with a coherent external electromagnetic field. No application whatsoever has been made of the rotating wave approximation. Being especially interested in the problem of higher harmonic generations for the case of very intense laser fields, we have developed in this Letter a much more efficient way to obtain these solutions as well as to carry out some calculations in a range in which the parameters take extreme values. Also the formalism allows us now to provide analytic expressions in the WKB regime for the electric dipole moment and the population inversion. The spectrum can be decomposed in periodic and non-periodic contributions. Only the latter depends upon the Floquet exponent and can be responsible of the main complexities of the observed Rabi revivals and the hyper-Raman shift. © 2002 Elsevier Science B.V. All rights reserved.

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In a previous paper [1] we have explored the appearance of higher harmonics as an effect arising from the monochromatic electric field of very intense laser interacting with a two-level atom. An adequate theoretical description of the interaction of the atom with the strong external field allow us, using the results presented in [2], to build up the time evolution operator of the system and exactly calculate the instantaneous atomic dipole moment and the population inversion. We obtain the Fourier transform of these quantities and identifying the present frequencies and its amplitudes we can explain the composition of the spectrum of the outgoing radiation. In the present Letter an entirely new viewpoint will be adopted shedding some light not only on the theoretical aspects already discussed but also enabling us to compute the correspondent physical quantities—either exactly or approximate—in a way much more accessible to experimental verification. In order to follow a self-contained approach we shall be presenting first a very brief account of [1].

Let us consider the physical system containing a *two-level atom* and its electric dipole interaction with the coherent field of a laser [1]. Let $|a\rangle$ and $|b\rangle$ be the atomic eigenstates with energies are 0 and $\hbar\omega_0$, respectively. The

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external monochromatic field frequency is ω and Ω_0 is the *Rabi frequency* corresponding to an interaction energy $\hbar\Omega_0$. In terms of the operators

$$J_0 = \frac{1}{2}\left\{|b\rangle\langle b| - |a\rangle\langle a|\right\}, \qquad J_- = |a\rangle\langle b|, \qquad J_+ = |b\rangle\langle a|, \tag{1}$$

the system can be described by the well known Hamiltonian

$$H(t) = \hbar\omega_0\left[J_0 + \frac{1}{2}\right] + \hbar\Omega_0[J_+ + J_-]\cos\omega t. \tag{2}$$

Here $H(t)$ is a Hermitian element of the triparametric $SU(2)$ Lie algebra and the physical states of the atom can be obtained by acting on the the eigenstates of J_0 with the time evolution operator (see Refs. [1,2])

$$U(t) = \exp\left\{-\frac{i\omega_0}{2}t\right\}\exp\left\{\eta(t)J_+\right\}\exp\left\{\gamma(t)J_0\right\}\exp\left\{-\eta^*(t)J_-\right\}\exp\left\{ih(t)J_0\right\}, \tag{3}$$

where $\gamma(t) = \mathrm{Ln}(1 + |\eta(t)|^2)$ and

$$h(t) = -\omega_0 t + 2\Omega_0\int_0^t \cos(\omega s)\,\mathrm{Re}\left[\eta(s)\right]ds, \tag{4}$$

$$\eta(t) = \frac{\omega_0 q(t) + 2i\dot{q}(t)}{\omega_0 q(t) - 2i\dot{q}(t)}. \tag{5}$$

The operator $U(t)$ satisfies the Schrödinger equation if the complex function $q(t)$ is a solution of the following *second-order ordinary linear differential equation* (ODE):

$$\ddot{q}(t) - 2i\Omega_0\cos(\omega t)\dot{q}(t) + \frac{\omega_0^2}{4}q(t) = 0, \qquad q(0) = 1, \qquad \dot{q}(0) = \frac{i\omega_0}{2}. \tag{6}$$

The main result that we would like to point out here is that it is in fact possible to condense the full dynamics of the physical problem in this linear differential equation [1]. Its solution determines all the mathematical quantities that will be considered of interest for the physical discussion of the laser–atom interaction. For instance, the rescaled dipole moment $D(t)$ and the population inversion $W(t)$ are defined from the complex solution $q(t) = r(t)\exp\{i\phi(t)\}$ of (6) as

$$D(t) = \left|q(t)\right|^2 - 1 = r^2(t) - 1, \qquad W(t) = -\frac{2}{\omega_0}r^2(t)\dot{\phi}(t). \tag{7}$$

Using (6) and (7) we can easily check that the following system of differential equations holds for $D(t)$ and $W(t)$:

$$\ddot{D}(t) + \omega_0^2 D(t) = 2\omega_0\Omega_0\cos(\omega t)W(t), \qquad D(0) = 0, \qquad \dot{D}(0) = 0, \tag{8}$$

$$\dot{W}(t) = -\frac{2\Omega_0}{\omega_0}\cos(\omega t)\dot{D}(t), \qquad W(0) = -1, \tag{9}$$

and after some manipulations the following fundamental invariant can easily be constructed:

$$\dot{D}^2(t) + \omega_0^2 D^2(t) + \omega_0^2 W^2(t) = \omega_0^2, \tag{10}$$

which guarantees a bound solution for any set of parameters ($\omega_0 \neq 0$). The differential system (8), (9) has already been discussed in the literature (see Refs. [3–5]). We have shown in [1] the relationship of this differential system with ODE (6), which has been proven to be crucial for establishing the non-trivial physical features to be discussed below. In particular, two important properties of this differential system must be emphasized: The existence of the first integral (10) and the direct relationship of $W(t)$ and $D(t)$ with a complex function $q(t) = r(t)\exp\{i\phi(t)\}$ which turns out to be a solution of an ordinary linear differential equation with periodic coefficients.

The description of the physical system in terms of $q(t)$ instead of the dipole moment $D(t)$ and the population inversion $W(t)$ is extremely advantageous as it allows us to express directly physical quantities such as the atomic radiation emitted by the excited atom, its spectral composition, the phases as well as the amplitudes of each component and the correspondent relationship with the relevant parameters of the system (i.e., the atomic frequency transition ω_0, the laser frequency ω and the Rabi frequency Ω_0). At this point we should mention the pioneer papers by Shirley [6], Zel'dovich [7], Cohen-Tannoudji and Haroche [8] and Eberly and coworkers [9] which can be considered the first serious attempts to solve this problem rigorously. There are, however, many aspects which have been either overlooked or treated in a different manner by these authors. The results hereby presented allows us to analyze in a much more efficient way other interesting effects. Among these we would like to mention the effect of the initial atomic state as well as that of the initial phase of the laser field with or without modulation terms. In order to achieve all these goals it seems essential to use a throughout analysis of the properties of $q(t)$ and hence those of $D(t)$ and $W(t)$ [4]. To this end we shall construct the Fourier spectrum of the relevant physical quantities. We can also present several approximate formulae for the phase and the amplitude for all present modes of the atomic emission spectra. Rescaling the time variable in the form $x=\omega t$ and using *primes* to denote derivatives with respect to x, the linear equation (6) reads

$$q''(x)-2i\gamma\cos x\, q'(x)+\frac{\epsilon^2}{4}q(x)=0,\qquad q(0)=1,\qquad q'(0)=i\frac{\epsilon}{2},\tag{11}$$

where the two new dimensionless parameters $\gamma=\Omega_0/\omega$ and $\epsilon=\omega_0/\omega$ are the ratio between the interaction and photon laser and the atomic and photon laser energies, respectively.

ODE (11) can be solved by applying Frobenius theory [1]. Let us define two independent functions $u(x)$ and $v(x)$ satisfying $u(0)=1$, $u'(0)=0$, $v(0)=0$, $v'(0)=1$. The Taylor coefficients are given in [1]. One can show that:

- The general solution of (11) is a linear superposition of $u(x)$ and $v(x)$. In particular, we set henceforth $q(x)=u(x)+i(\epsilon/2)v(x)$.
- The equation possesses a *first integral* $|q'|^2+(\epsilon^2/4)|q|^2=C_0$ which allows us to set some bounds for the solutions $u(x)$ and $v(x)$ and yields as a consequence of the *first integral* (11) the absolute bound:

$$\left|u^2(x)\right|+\frac{\epsilon^2}{4}\left|v^2(x)\right|=1.\tag{12}$$

- The set of functions $u(x)$ and $v(x)$ also satisfy

$$u'(x)=-\frac{\epsilon^2}{4}e^{2i\gamma\sin x}v^*(x),\qquad v'(x)=e^{2i\gamma\sin x}u^*(x).\tag{13}$$

  These relationships clearly show that $u(x)$ is 2π-periodic (2π-antiperiodic) if and only if $v(x)$ exhibits also the same properties. Thus, for any set of parameters ($\epsilon\neq 0$) for which a solution of (11) exists with the property of being 2π-periodic (2π-antiperiodic), we can claim that all solutions will also be 2π-periodic (2π-antiperiodic). Notice also that $u(x)$ and $v(x)$ have the following symmetry properties:

$$u(x+\pi)=u(\pi)u^*(x)-\frac{\epsilon^2}{4}v^*(\pi)v^*(x),\tag{14}$$

$$v(x+\pi)=v(\pi)u^*(x)+u^*(\pi)v^*(x).\tag{15}$$

  Also $u(-x)=u^*(x)$ and $v(-x)=-v^*(x)$. Clearly, these properties can be used to know the values of the functions $u(x)$ and $v(x)$ for all x if they are known just on the $[0,\pi]$ interval. This interval may be reduced in fact to $[0,\pi/2]$. In particular, $(1/2)(u(2\pi)+v'(2\pi))=\mathrm{Re}[u(2\pi)]=1-2(\epsilon\,\mathrm{Re}[u(\pi/2)v^*(\pi/2)])^2$.

- The set of functions $D(x)$ and $W(x)$ may be expressed as

$$D(x) = \epsilon\,\mathrm{Im}\big(u(x)v^*(x)\big), \qquad W(x) = -\,\mathrm{Re}\left[e^{-2i\gamma\sin x}\left(u^2(x) + \frac{\epsilon^2}{4}v^2(x)\right)\right]. \tag{16}$$

- As Eq. (11) has periodic coefficients a Floquet analysis can be performed. According to this theorem any solution can be expressed in the form

$$q(x) = Ae^{i\nu x}F_+(x) + Be^{-i\nu x}F_-(x), \tag{17}$$

where $F_+(x)$ and $F_-(x)$ are 2π-periodic functions. Using the above properties we end up with an explicit expression for the Floquet exponent, namely:

$$\nu = \frac{1}{2\pi}\arccos\big[\mathrm{Re}\big[u(2\pi)\big]\big] = \frac{1}{\pi}\arcsin\left[\epsilon\,\mathrm{Re}\left|u\left(\frac{\pi}{2}\right)v^*\left(\frac{\pi}{2}\right)\right|\right], \quad 0 \leqslant \nu \leqslant \frac{1}{2}. \tag{18}$$

Its physical significance lies on the fact that it actually determines the spectral decomposition of the emitted atomic radiation. The two extreme cases correspond to solutions 2π-periodic ($\nu = 0$) or 2π-antiperiodic ($\nu = 1/2$).

The Fourier spectrum of $u(x)$ and $v(x)$ takes the following generic form:

$$f(x) = A\sum_{j=-\infty}^{j=\infty} F_j e^{i(j+\nu)x} + B\sum_{j=-\infty}^{j=\infty} (-1)^j F_{-j} e^{i(j-\nu)x}, \tag{19}$$

where the *Floquet exponent* ν and the *Fourier coefficients* F_j can easily be found simultaneously with the help of the following recurrence relation:

$$\frac{1}{\gamma}\left\{(p+\nu)^2 - \frac{\epsilon^2}{4}\right\}F_p = (p+\nu+1)F_{p+1} + (p+\nu-1)F_{p-1}, \tag{20}$$

which can be solved through a numerical method using *continued fractions* [1]. The instantaneous dipole moment of the atom can finally be expressed as

$$D(x) = \epsilon\,\mathrm{Im}\big(u(x)v^*(x)\big) = \Delta_1(x) + \Delta_2(x). \tag{21}$$

This expression is in fact the sum of two different contributions. The $\Delta_1(x)$ term is a superposition of odd harmonics of the laser frequency. The $\Delta_2(x)$ term is a superposition of even harmonics shifted up and down an amount given by 2ν:

$$\Delta_1(x) = \sum_{j=0}^{\infty} D_j \cos\big[(2j+1)x\big], \tag{22}$$

$$\Delta_2(x) = \sum_{j=0}^{\infty} D_j^+ \cos\big[2(j+\nu)x\big] + \sum_{j=1}^{\infty} D_j^- \cos\big[2(j-\nu)x\big]. \tag{23}$$

The correspondent Fourier amplitudes are

$$D_j = 2\frac{M_e^2 - M_o^2}{(M_e^2 + M_o^2)^2}\sum_{r=-\infty}^{\infty} F_r F_{r+2j+1}, \quad M_e = \sum_{r=-\infty}^{\infty} F_{2r}, \tag{24}$$

$$D_j^{\pm} = -2\frac{M_e M_o}{(M_e^2 + M_o^2)^2}\sum_{r=-\infty}^{\infty} (-1)^r F_{-r} F_{r\pm 2j}, \quad M_o = \sum_{r=-\infty}^{\infty} F_{2r+1}. \tag{25}$$

The Fourier spectrum of $D(x)$ dictates the composition of the emitted atomic radiation in interaction with a laser. A qualitative picture shows the appearance of a triplet centered in the odd harmonics with frequencies $(2s+1\pm\delta)\omega$. The amount of the shift δ is given by the Floquet exponent (18) in the form $\delta = 1 - 2\nu$ and coincides with the generalized Rabi frequency. The non-periodic contribution $\Delta_2(x)$ of the instantaneous atomic dipole moment, which can be interpreted as a superposition of odd harmonics shifted up and down an amount δ, is the origin of the hyper-Raman peaks of the triplet. Likewise, the population inversion can also be cast in the form

$$W(x) = -1 - 2\frac{\gamma}{\epsilon}\int_0^x \cos z\, D'(z)\,dz = \Pi_1(x) + \Pi_2(x) \tag{26}$$

as the sum of two contributions $\Pi_1(x)$ (a superposition of even harmonics of the laser frequency) and $\Pi_2(x)$ which is a superposition of odd harmonics shifted up and down an amount given by 2ν:

$$\Pi_1(x) = \sum_{j=0}^{\infty} W_j \cos[2jx], \tag{27}$$

$$\Pi_2(x) = \sum_{j=1}^{\infty} W_j^+ \cos\bigl[(2j+1+2\nu)x\bigr] + \sum_{j=0}^{\infty} W_j^- \cos\bigl[(2j+1-2\nu)x\bigr], \tag{28}$$

with Fourier amplitudes given by

$$W_0 = -\bigl(1 + W_0^-\bigr) - \sum_{j=1}^{\infty}\bigl(W_j + W_j^+ + W_j^-\bigr), \tag{29}$$

$$W_j = -\frac{\gamma}{\epsilon}\left(D_j + D_{j-1} + \frac{D_j - D_{j-1}}{2j}\right), \quad j \geqslant 1, \tag{30}$$

$$W_0^- = -\frac{\gamma}{\epsilon}\left(D_0^+ + D_1^- + \frac{D_1^- - D_0^+}{1-2\nu}\right), \tag{31}$$

$$W_j^\pm = -\frac{\gamma}{\epsilon}\left(D_j^\pm + D_{j+1}^\pm + \frac{D_{j+1}^\pm - D_j^\pm}{2j+1\pm 2\nu}\right), \quad j \geqslant 1. \tag{32}$$

Aside from the numerical procedure provided by (18), (24), (25) and (29)–(32), a complete analytical reconstruction of the spectrum can be carried out starting with an input provided by the $D(x)$ and $W(x)$ functions. If one can find analytic expressions for the functions $u(x)$ and $v(x)$, the values for $D(x)$ and $W(x)$ are immediately known from (16) and the amplitudes D_j, $D_j^\pm$, W_j, and $W_j^\pm$ can easily be found as

$$D_j = \frac{4}{\pi}\int_0^{\pi/2} \Delta_1(x)\cos\bigl[(2j+1)x\bigr]\,dx, \tag{33}$$

$$D_j^\pm = \frac{2}{\pi}\int_0^{\pi/2} \Delta_2(x)\cos\bigl[(2j\pm\nu)x\bigr]\,dx \pm \frac{1}{\pi\sin(2\pi\nu)}\int_0^{\pi/2}\bigl\{\Delta_2(x-\pi) - \Delta_2(x+\pi)\bigr\}\sin\bigl[(2j\pm\nu)x\bigr]\,dx, \tag{34}$$

$$W_j = \frac{2m_j}{\pi}\int_0^{\pi/2} \Pi_1(x)\cos[2jx]\,dx, \tag{35}$$

$$W_j^{\pm} = \frac{2}{\pi} \int_0^{\pi/2} \Pi_2(x) \cos\bigl[(2j+1 \pm 2\nu)x\bigr]\,dx$$
$$\mp \frac{1}{\pi \sin(2\pi\nu)} \int_0^{\pi/2} \bigl\{\Pi_2(x-\pi) - \Pi_2(x+\pi)\bigr\} \sin\bigl[(2j+1 \pm 2\nu)x\bigr]\,dx, \tag{36}$$

where $m_0 = 1$ and $m_j = 2$ $(j \geqslant 1)$. These expressions are obviously exact as long as so were the knowledge of the Floquet exponent ν and the functions $u(x)$ and $v(x)$.

As a concrete example of the above discussion, we shall be considering the case in which the parameters of the physical system verify $\gamma \approx \epsilon \gg 1$. This condition holds for values of the laser frequency smaller than the values of the resonance frequency and the coupling constant $\alpha = \gamma/\epsilon$ is of the order of one. In this way we are considering subresonant systems $\omega \ll \omega_0 \approx \Omega_0$ with interaction and transition energies of similar values. Since in other papers [4,5] a numerical study of this range of parameters has been made we have chosen the same range for the sake of comparison. This case seems also quite adequate for using the WKB method for high values of the parameter ϵ. In regard to Eqs. (11)–(13) this means that we are looking for a solution of in the form $q(x) = \exp\{i\gamma \sin x + \epsilon \sum_{k=0}^{\infty} z_k(x)\epsilon^{-k}\}$, where the $z_k(x)$ verify

$$z_0'^2(x) = -\frac{1}{4}\bigl(1 + 4\alpha^2 \cos^2 x\bigr), \tag{37}$$

$$z_k'(x) = -\frac{1}{2z_0'(x)} \left\{ z_{k-1}''(x) + \sum_{j=1}^{k-1} z_j'(x) z_{k-j}'(x) \right\}, \quad k \geqslant 1. \tag{38}$$

After calculating the sequence for higher orders of approximation, $u(x)$ and $v(x)$ can be expressed as

$$u(x) \approx \frac{\sqrt{\lambda} e^{i\gamma \sin x}}{2(1 + 4\alpha^2 \cos^2 x)^{1/4}} \left\{ \left(1 - \frac{2\alpha}{\lambda}\right) f_1(\alpha, \epsilon, x) + \left(1 + \frac{2\alpha}{\lambda}\right) f_2(\alpha, \epsilon, x) \right\}, \tag{39}$$

$$v(x) \approx \frac{i e^{i\gamma \sin x}}{\epsilon\sqrt{\lambda}} \frac{\{f_2(\alpha, \epsilon, x) - f_1(\alpha, \epsilon, x)\}}{(1 + 4\alpha^2 \cos^2 x)^{1/4}}, \tag{40}$$

$$f_1(\alpha, \epsilon, x) \approx \bigl[(\lambda + 2\alpha)\bigl(\sqrt{1 + 4\alpha^2 \cos^2 x} - 2\alpha \cos x\bigr)\bigr]^{1/2} \Phi(\alpha, \epsilon, x), \tag{41}$$

$$f_2(\alpha, \epsilon, x) \approx \bigl[(\lambda - 2\alpha)\bigl(\sqrt{1 + 4\alpha^2 \cos^2 x} + 2\alpha \cos x\bigr)\bigr]^{1/2} \Phi^*(\alpha, \epsilon, x), \tag{42}$$

$$\Phi(\alpha, \epsilon, x) \approx \exp\left\{ i \left(\frac{\epsilon\lambda E[x, k^2]}{2} + \frac{(1 + 8\alpha^2)E[x, k^2] - F[x, k^2]}{12\epsilon\lambda} \right) \right\}, \tag{43}$$

where $\lambda = \sqrt{1 + 4\alpha^2}$, $k^2 = 4\alpha^2/(1 + 4\alpha^2)$, and $F[x, k^2]$ and $E[x, k^2]$ are the incomplete elliptic integrals of the first and second kind, respectively. From these expressions one can derive the values of the Floquet exponent as well as the values of the instantaneous dipole moment. These values take the form

$$\Omega(\epsilon, \alpha) = \frac{1}{\pi} \left(\epsilon\lambda E\bigl[k^2\bigr] + \frac{(1 + 8\alpha^2)E[k^2] - K[k^2]}{6\epsilon\lambda} \right), \tag{44}$$

$$\nu(\epsilon, \alpha) \approx \arcsin\bigl|\sin \Omega(\epsilon, \alpha)\bigr|, \quad 0 \leqslant \nu \leqslant \frac{1}{2}. \tag{45}$$

For sufficiently large values of ϵ and practically *any value* of α $(\epsilon \gg \alpha)$ the Floquet exponent ν is a $2\epsilon_0$-periodic function of ϵ of the form

$$\nu \approx \frac{\epsilon}{2\epsilon_0} \quad \text{if } 0 \leqslant \epsilon \leqslant \epsilon_0, \tag{46}$$

$$\nu \approx 1 - \frac{\epsilon}{2\epsilon_0} \quad \text{if } \epsilon_0 \leqslant \epsilon \leqslant 2\epsilon_0, \tag{47}$$

where

$$\epsilon_0 = \pi \left(2\sqrt{1+4\alpha^2}\, E\left[\frac{4\alpha^2}{1+4\alpha^2} \right] \right)^{-1}$$

and ν repeats itself every time that ϵ increases in $2\epsilon_0$. For a given value of the coupling these oscillations do not die away and keep going even if the laser frequency decreases. In particular, when $\alpha = 1$ (the energy of interaction equals the energy of the transition) the value of ϵ_0 is approximately 0.596086. Notice the predictive power of (44) and (45). One can use it for selecting values of ϵ and α with a given Floquet exponent ν. Thus the spectral composition of the dipole moment and the population inversion can be selected at will with a judicious choice of the parameters. Finally one can easily conclude that

$$\epsilon \approx \frac{r\pi}{2\sqrt{1+4\alpha^2}\, E[k^2]} \left\{ 1 + \sqrt{1 - \frac{2E[k^2]((1+8\alpha^2)E[k^2] - K[k^2])}{3r^2\pi^2}} \right\}. \tag{48}$$

If $r \approx n$, $\nu = 0$, and the functions $u(x)$ and $v(x)$ are 2π-periodic. Both the dipole moment $D(x)$ and the population inversion $W(x)$ are composed by pure even and odd harmonics of the laser. If $r \approx n + 1/2$, $\nu = 1/2$, and $u(x)$ and $v(x)$ are 2π-antiperiodic. The spectrum of $D(x)$ contains just odd harmonics and that of $W(x)$ just even harmonics. If $r \approx (1/2)(n + 1/2)$, $\nu = 1/4$, and the functions $u(x)$ and $v(x)$ are 4π-antiperiodic. The atomic dipole moment can be constructed in terms of its following two components:

$$\Delta_1(\alpha, x) = -\frac{2\alpha}{\lambda} \frac{\cos x}{\sqrt{1+4\alpha^2 \cos^2 x}}, \tag{49}$$

$$\Delta_2(\alpha, \epsilon, x) = \frac{2\alpha}{\lambda} \frac{\cos\{\epsilon\lambda E[x, k^2] + (1/6\epsilon\lambda)\{(1+8\alpha^2)E[x, k^2] - F[x, k^2]\}\}}{\sqrt{1+4\alpha^2 \cos^2 x}}. \tag{50}$$

Notice that $\Delta_1(\alpha, x)$ is independent of ϵ and it is a purely even 2π-periodic term under the periodic translation. Therefore, this term enjoys the adequate symmetry to represent the periodic part of the dipole oscillation. One can easily identify a term like this as an oscillation at the laser frequency, with modulated amplitude and just composed by odd harmonics of this frequency. The term $\Delta_2(\alpha, \epsilon, x)$ has quite different properties being periodic just for those values of the parameters for which the Floquet exponent ν takes rational values. The values of the instantaneous dipole moment hereby found can be used to find its respective spectral components. The periodic part yields the odd harmonics spectrum with amplitude and phase are given by

$$D_j = -\frac{8\alpha}{\pi\sqrt{1+4\alpha^2}} \int_0^{\pi/2} \frac{\cos x \cos[(2j+1)x]}{\sqrt{1+4\alpha^2 \cos^2 x}}, \quad j \geqslant 0. \tag{51}$$

For sufficiently lower laser frequencies (sufficiently large ϵ) the odd harmonic spectrum results quite independent of this frequency. The different amplitudes depend just on the strength of the laser–atom interaction and smoothly decrease for higher-order harmonics. In particular, the amplitude and phase of the first harmonic can explicitly be given by

$$D_0 = -\frac{2}{\pi\alpha} \left(E\left[\frac{4\alpha^2}{1+4\alpha^2} \right] - \frac{1}{1+4\alpha^2} K\left[\frac{4\alpha^2}{1+4\alpha^2} \right] \right). \tag{52}$$

The second contribution gives rise to the hyper-Raman spectrum. For any set of the physical parameters of the system, either $\Omega+\nu$ or $\Omega-\nu$ must be an integer number. One can construct the correspondent amplitudes in the form

$$D_j^{+}=\frac{4\alpha}{\pi\lambda}\int_0^{\pi/2}\frac{\cos[(\epsilon\lambda+(1+8\alpha^2)/6\epsilon\lambda)E[x,k^2]-(1/6\epsilon\lambda)K[x,k^2]\pm 2(j+\nu)x]}{\sqrt{1+4\alpha^2\cos^2 x}},\quad j\geqslant 0, \tag{53}$$

$$D_j^{-}=\frac{4\alpha}{\pi\lambda}\int_0^{\pi/2}\frac{\cos[(\epsilon\lambda+(1+8\alpha^2)/6\epsilon\lambda)E[x,k^2]-(1/6\epsilon\lambda)K[x,k^2]\mp 2(j-\nu)x]}{\sqrt{1+4\alpha^2\cos^2 x}},\quad j\geqslant 1, \tag{54}$$

where the upper sign holds for $\Omega+\nu$ an integer and the lower sign for the integer being $\Omega-\nu$. A large number of oscillations substantially lowers the value of the integral, the correspondent values of the amplitudes leads one to consider easily the region of the spectrum of maximal amplitudes. Therefore, for the first case, the amplitude of the harmonics with frequencies $2j+2\nu$ will be negligible while the harmonics with frequencies $2j-2\nu$ will have significant amplitudes which can even have similar intensities to that of the odd harmonics. These large amplitudes will be distributed over a region with values of $2j$ close to $\Omega+\nu$. This behaviour justifies the typical harmonic generation spectrum found by experiment showing an intensity distribution that first decreases rather steeply as the harmonic order increases (the odd spectrum) and then after remaining almost flat for a number of harmonics finally decreases steeply again in a “plateau” form (the non-integral spectrum) [10]. All these facts are in fairly good agreement with the cases discussed in [3] where the shift δ is given by an elliptic integral as in our case, but the correspondent intensities are given in terms of Bessel functions. Likewise, the population inversion can explicitly be characterized by using the two contributions

$$\Pi_1(\alpha,x)=-\frac{1}{\lambda\sqrt{1+4\alpha^2\cos^2 x}}, \tag{55}$$

$$\Pi_2(\alpha,\epsilon,x)=-\frac{4\alpha^2}{\lambda}\frac{\cos x\cos[(\epsilon\lambda+(1+8\alpha^2)/6\epsilon\lambda)E[x,k^2]-(1/6\epsilon\lambda)F[x,k^2]]}{\sqrt{1+4\alpha^2\cos^2 x}}, \tag{56}$$

and as in the previous case $\Pi_1(\alpha,x)$ is independent of ϵ. It is a purely π-periodic term with the adequate symmetry to represent the periodic part of the population inversion whose Fourier spectrum is just composed by even harmonics of the laser frequency with amplitudes

$$W_j=-\frac{2m_j}{\pi\sqrt{1+4\alpha^2}}\int_0^{\pi/2}\frac{\cos[2jx]}{\sqrt{1+4\alpha^2\cos^2 x}}, \tag{57}$$

where $m_0=1$ and $m_j=2$ ($j\geqslant 1$). The term $\Pi_2(\alpha,\epsilon,x)$ depends on the two characteristic parameters of the system and yields the rest of the spectrum with amplitudes

$$W_j^{+}=-\frac{8\alpha^2}{\pi\lambda}\int_0^{\pi/2}\frac{\cos x\cos[(\epsilon\lambda+(1+8\alpha^2)/6\epsilon\lambda)E[x,k^2]-(1/6\epsilon\lambda)K[x,k^2]\pm(2j+1+2\nu)x]}{\sqrt{1+4\alpha^2\cos^2 x}},\quad j\geqslant 0, \tag{58}$$

$$W_j^{-}=-\frac{8\alpha^2}{\pi\lambda}\int_0^{\pi/2}\frac{\cos x\cos[(\epsilon\lambda+(1+8\alpha^2)/6\epsilon\lambda)E[x,k^2]-(1/6\epsilon\lambda)K[x,k^2]\mp(2j+1-2\nu)x]}{\sqrt{1+4\alpha^2\cos^2 x}},\quad j\geqslant 1, \tag{59}$$

with the same convention of signs as the one given above.

Conclusions. In a previous paper [1], the present authors had already considered the system composed by a two-level atom interacting with a coherent external electromagnetic field by solving the main equations in the Schrödinger picture without using the rotating wave approximation. We were specially interested in the spectral composition of the atomic dipole moment and the population inversion. These quantities are determined by means of two functions $u(x)$ and $v(x)$ which are independent solutions of an ordinary differential equation with periodic coefficients. Floquet analysis applied to this equation yields the frequencies which are present in the spectra and allows us to identify two different contributions in $D(x)$ (the atomic dipole moment) and $W(x)$ (the population inversion) which yield the different spectral components through quadratures. In this Letter we have presented *analytical expressions for the frequencies and amplitudes within a particular range of values of the physical parameters*. These expressions may be used to analyze the influence of the different parameters in the spectra. Our results are also clearly of experimental interest in assigning odd and Floquet-shifted character to each observed harmonic. This gives rise to the decreasing and flat parts of the observed spectrum.

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