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SEPARATION OF VARIABLES IN A NONLINEAR WAVE EQUATION WITH A VARIABLE WAVE SPEED

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We develop a generalized conditional symmetry approach for the functional separation of variables in a nonlinear wave equation with a nonlinear wave speed. We use it to obtain a number of new (1+1)*-dimensional nonlinear wave equations with variable wave speeds admitting a functionally separable solution. As a consequence, we obtain exact solutions of the resulting equations.*

Keywords: Lie symmetries, generalized symmetries, diffusion equations, nonlinear equations

The classical theory of symmetries of differential equations due to Lie [1], [2] is a powerful tool for studying the separation of variables of linear partial differential equations [3]. Recently, two of the more interesting extensions of the Lie theory, the nonclassical method [4] and the generalized conditional symmetry (GCS) approach [5]–[7], have been used to study the functional separation of variables of nonlinear diffusion equations with convection and source terms [8], [9]. Another interesting example is the nonlinear wave equation with a variable wave speed,

$$u_{tt} = (B(u)u_x)_x + A(u), \qquad B(u) \neq \text{const}, \tag{1}$$

which has significant applications in wave propagation and applied sciences.

In this paper, we use the GCS method to study the functional separation of variables of Eq. (1). A solution of (1) is said to be functionally separable if there exist functions $q(u)$, $\phi(t)$, and $\psi(x)$ such that

$$q(u) = \phi(t) + \psi(x). \tag{2}$$

The classical additively separable solution and product separable solution are particular cases of the above functional separable solution. In the case where $B = 1$, the functional separation of variables in Eq. (1) was discussed by several authors using different kinds of methods [10]. Equations admitting separable solutions include the Bullough–Dodd, sine-Gordon, and sinh-Gordon equations. The case where $A = 0$ with $B = e^{au}$ or $B = u^a$ was discussed in [11] using the classical Lie method.

Definition. An evolutionary vector field

$$V = \eta(t, x, u, \dots)\frac{\partial}{\partial u} \tag{3}$$

is said to be a GCS of (1) if

$$V^{(2)}\big(u_{tt} - (B(u)u_x)_x - A(u)\big)\big|_{E\cap W} = 0,$$

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where E is the solution manifold of (1) and W is a second-order system of (1) obtained by appending the condition $\eta = 0$ and its partial derivatives with respect to x to the invariant surface; $V^{(2)}$ is the second prolongation of the infinitesimal operator V.

An important fact is that if (1) admits the GCS,

$$V = (u_{xt} + G(u)u_x u_t)\frac{\partial}{\partial u}, \qquad G = \frac{q''(u)}{q'(u)}, \tag{4}$$

then (1) has functionally separable solution (2).

A straightforward calculation gives

$$\begin{aligned} V^{(2)}\big(u_{tt} - (B(u)u_x)_x - A(u)\big)\big|_{E\cap W} = \\ = 3\left(B'' - GB' - \frac{B'^2}{B}\right)u_t u_x u_{xx} + \left[G'' - 2GG' - \frac{B'}{B}(G' - G^2)\right]u_x u_t^3 + \\ + \left[B''' - \frac{B'}{B}(B'' + GB') - B(G'' - 2GG') - GB'' - G^2B'\right]u_t u_x^3 + \\ + \left[A'' + \left(G - \frac{B'}{B}\right)A' + \left(3G' - 2G^2 - \frac{B'}{B}G\right)A\right]u_t u_x. \end{aligned}$$

The vanishing of this expression leads to B, G, and A satisfying

$$B'' - GB' - \frac{B'^2}{B} = 0, \tag{5a}$$

$$G'' - 2GG' - \frac{B'}{B}(G' - G^2) = 0, \tag{5b}$$

$$A'' + \left(G - \frac{B'}{B}\right)A' + \left(3G' - 2G^2 - \frac{B'}{B}G\right)A = 0. \tag{5c}$$

We consider two cases for Eq. (5b):

Case 1. $G' - G^2 = 0$. In this case, G is given by

$$G = 0 \quad \text{or} \quad G = -\frac{1}{u + u_0},$$

where u_0 can be chosen to be zero by translating u. The classical additively separable solution is given by $G = 0$; $G = -1/u$ then gives the product separable solution. We consider each case separately.

Case 1a. $G = -1/u$. Substituting $G = -1/u$ in (5a) and (5c) and scaling u, we find

$$B = u^a, \quad a = \text{const},$$

and A satisfying

$$A'' - \frac{a+1}{u}A' + \frac{a+1}{u^2}A = 0,$$

which is solved by

$$A = c_1 u + c_2 u^{a+1},$$

where c_1 and c_2 are arbitrary constants (here and hereafter). Furthermore, from (4), we have $q = \log u$. We have thus established that the equation

$$u_{tt} = (u^a u_x)_x + c_1 u + c_2 u^{a+1} \tag{6}$$

admits the product separable solution

$$u = \phi(t)\psi(x), \tag{7}$$

where $\phi(t)$ and $\psi(x)$ satisfy the system

$$\phi'' - c_1\phi - \lambda\phi^{a+1} = 0, \tag{8a}$$

$$(\psi^a\psi')' + c_2\psi^{a+1} - \lambda\psi = 0, \tag{8b}$$

and λ denotes the separation constant. Equation (8a) can be integrated as

$$\begin{aligned} &\phi'^2 - c_1\phi^2 - \frac{2\lambda}{a+2}\phi^{a+2} = d_1 \quad && \text{if } a \neq -2, \\ &\phi'^2 - c_1\phi^2 - 2\lambda\log\phi = d_1 && \text{if } a = -2. \end{aligned}$$

The first integral of Eq. (8b) is

$$\begin{aligned} &\psi'^2 + \frac{c_2}{a+1}\psi^2 - \frac{2\lambda}{a+2}\psi^{2-a} = d_2\psi^{-2a} \quad && \text{if } a \neq -2, \quad a \neq -1, \\ &\psi'^2 - c_2\psi^2 - 2\lambda\psi^4\log\psi = d_2\psi^4 && \text{if } a = -2, \\ &\psi'^2 + 2c_2\psi^2\log\psi - 2\lambda\psi^3 = d_2\psi^2 && \text{if } a = -1. \end{aligned}$$

Consequently, the solutions for ϕ and ψ can be written implicitly as follows:

1a.1. If $a \neq -2, -1$,

$$\int^{\phi(t)} \frac{dz}{\sqrt{2\lambda z^{2+a}/(a+2) + c_1 z^2 + d_1}} = t,$$

$$\int^{\psi(x)} \frac{dy}{\sqrt{2\lambda y^{2-a}/(a+2) - c_2 y^2/a+1 + d_2 y^{-2a}}} = x.$$

1a.2. If $a = -2$,

$$\int^{\phi(t)} \frac{dz}{\sqrt{c_1 z^2 + 2\lambda\log z + d_1}} = t,$$

$$\int^{\psi(x)} \frac{dy}{\sqrt{2\lambda y^4\log y + c_2 y^2 + d_2 y^4}} = x.$$

1a.3. If $a = -1$,

$$\int^{\phi(t)} \frac{dz}{\sqrt{c_1 z^2 + 2\lambda z + d_1}} = t,$$

$$\int^{\psi(x)} \frac{dy}{\sqrt{2\lambda y^3 - 2c_2 y^2\log y + d_2 y^2}} = x.$$

Case 1b. $G = 0$. In this subcase,

$$B = e^u, \qquad A = c_1 + c_2 e^u, \qquad q = u.$$

We find that the equation

$$u_{tt} = (e^u u_x)_x + c_1 + c_2 e^u \tag{9}$$

admits the additively separable solution

$$u = \phi(t) + \psi(x), \tag{10}$$

where $\phi(t)$ and $\psi(x)$ satisfy the system

$$\begin{aligned} &\phi'' - c_1 - \lambda e^{\phi} = 0, \\ &(e^{\psi}\psi')' + c_2 e^{\psi} - \lambda = 0, \end{aligned} \tag{11}$$

which can be integrated as follows:

1b.1. If $c_2 > 0$,

$$\int^{\phi(t)} \frac{dz}{\sqrt{2\lambda e^z + 2c_1 z + d_1}} = t,$$

$$\psi(x) = \log\left[\frac{\lambda}{c_2} - d_2 \cos\sqrt{c_2}x\right].$$

1b.2. If $c_2 < 0$,

$$\int^{\phi(t)} \frac{dz}{\sqrt{2\lambda e^z + 2c_1 z + d_1}} = t,$$

$$\psi(x) = \log\left[\frac{\lambda}{c_2} - d_2 \cosh\sqrt{-c_2}x\right].$$

Case 2. $G' - G^2 \neq 0$. We can define $h(u)$ such that

$$G = -\frac{h'}{h} \quad \Longrightarrow \quad q' \sim \frac{1}{h}.$$

From (5b), we find

$$B = \frac{B_0 h''}{h}, \tag{12}$$

where B_0 is a constant and can be chosen as ± 1 by scaling t. Substituting $G = -h'/h$ in (5a), we also find $B'/B = a/h$ and $a = \text{const}$, and B can hence be expressed in terms of h by

$$\frac{B'}{B} = \frac{a}{h}. \tag{13}$$

Obviously, h satisfies

$$hh''' - h'h'' = ah'',$$

which can be integrated as

$$hh'' - h'^2 = ah' + b. \tag{14}$$

We note that we explicitly assume that $a \neq 0$ in what follows because it is easy to see from Eq. (13) that $a = 0 \Longrightarrow B = \text{const}$, which corresponds to the case extensively considered by other authors in [10], [11].

For $a \neq 0$, two subcases arise:

Case 2a. $b = 0$. Solving (14), we obtain

$$h = \frac{a}{d} + ce^{du},$$

where d is an arbitrary constant. Hence,

$$B = -\frac{B_0 d^2 e^{du}}{a/(dc) + e^{du}}.$$

By scaling and translating u, we can set $d = \pm 1$ and $a/c = \pm 1$. Four possibilities are distinguished:

2a.1. $B_0 = -d = 1$, $a/c = 1$. In this case,

$$B = \frac{1}{e^u - 1}, \qquad h = c(e^{-u} - 1), \qquad q \sim \log(e^u - 1).$$

Substituting B and h in (5c) implies that A satisfies the equation

$$A'' + A' + \frac{2}{e^u - 1} A = 0,$$

which has the general solution

$$A(u) = c_1 \big[1 - 2e^{-u} + 2(e^{-u} - e^{-2u}) \log(e^u - 1)\big] + c_2(e^{-2u} - e^{-u}).$$

We have thus established that the equation

$$u_{tt} = \left(\frac{u_x}{e^u - 1}\right)_x + c_1 \big[1 - 2e^{-u} + 2(e^{-u} - e^{-2u}) \log(e^u - 1)\big] + c_2(e^{-2u} - e^{-u}) \tag{15}$$

admits the functionally separable solution

$$u = \log[1 + \phi(t)\psi(x)]. \tag{16}$$

Substituting (16) in (15) implies that $\phi(t)$ and $\psi(x)$ satisfy the system

$$\begin{aligned} \phi'^{\,2} &= 2c_1 \phi^2 \log \phi + (\alpha - c_1)\phi^2 - 2\lambda\phi - \beta, \\ \psi'^{\,2} &= 2c_1 \psi^2 \log \psi + \beta\psi^4 - 2\lambda\psi^3 + (c_1 - c_2 - \alpha)\psi^2, \end{aligned} \tag{17}$$

where λ, α, and β are arbitrary constants. The implicit solution of (17) is given by

$$\begin{aligned} \int^{\phi(t)} \frac{dz}{\sqrt{2c_1 z^2 \log z + (\alpha - c_1) z^2 - 2\lambda z - \beta}} &= t, \\ \int^{\psi(x)} \frac{dy}{\sqrt{2c_1 y^2 \log y + \beta y^4 - 2\lambda y^3 + (c_1 - c_2 - \alpha) y^2}} &= x. \end{aligned} \tag{18}$$

The remaining cases are studied similarly, we list the results as follows:

2a.2. $B_0 = -d = -1$, $a/c = 1$.

$$B = \frac{e^u}{1+e^u}, \qquad h = c(1+e^u), \qquad q = u - \log(1+e^u).$$

The resulting equation is

$$u_{tt} = \left(\frac{e^u}{e^u+1}u_x\right)_x - c_1\left[1 + 2e^u + 2(e^u+e^{2u})\log\frac{e^u}{e^u+1}\right] + c_2(e^{2u}+e^u), \tag{19}$$

and the functionally separable solution is

$$u = \log\left[\frac{1}{\phi(t)\psi(x)-1}\right]. \tag{20}$$

The ordinary differential equations for ϕ and ψ are

$$\begin{aligned} \phi'^{\,2} &= 2c_1\phi^2\log\phi + (\alpha - c_1)\phi^2 - 2\lambda\phi - \beta, \\ \psi'^{\,2} &= -2c_1\psi^2\log\psi - \beta\psi^4 - 2\lambda\psi^3 + (\alpha - c_1 - c_2)\psi^2, \end{aligned} \tag{21}$$

which can be integrated as

$$\begin{aligned} &\int^{\phi(t)} \frac{dz}{\sqrt{2c_1z^2\log z + (\alpha-c_1)z^2 - 2\lambda z - \beta}} = t, \\ &\int^{\psi(x)} \frac{dy}{\sqrt{-2c_1y^2\log y - \beta y^4 - 2\lambda y^3 + (\alpha - c_1 - c_2)y^2}} = x. \end{aligned} \tag{22}$$

2a.3. $B_0 = -d = -1$, $a/c = -1$.

$$B = \frac{e^u}{e^u-1}, \qquad h = c(e^u-1), \qquad q = -u + \log(e^u-1),$$

$$u_{tt} = \left(\frac{e^u}{e^u-1}u_x\right)_x + c_1\left[2e^u - 1 - 2(e^{2u}-e^u)\log\frac{e^u}{e^u-1}\right] + c_2(e^{2u}-e^u), \tag{23}$$

$$u = \log\left[\frac{1}{1-\phi(t)\psi(x)}\right]. \tag{24}$$

The equations for ϕ and ψ are also Eqs. (21). Therefore, the solutions for ϕ and ψ are given by (22).

2a.4. $B_0 = d = -1$, $a/c = -1$.

$$B = \frac{1}{1+e^u}, \qquad h = c(1+e^{-u}), \qquad q = \log(1+e^u),$$

$$u_{tt} = \left(\frac{u_x}{e^u+1}\right)_x + c_1\left[2e^{-u} + 1 - 2(e^{-2u}+e^{-u})\log(e^u+1)\right] - c_2(e^{-2u}+e^{-u}), \tag{25}$$

$$u = \log\left[\phi(t)\psi(x) - 1\right]. \tag{26}$$

As in the previous case, ϕ and ψ must satisfy Eqs. (21), and its solutions are implicitly given by (22).

Case 2b. $b \neq 0$. With $H(h) = h_u$ introduced in (6), H satisfies

$$hHH_h = H^2 + aH + b,$$

which is integrated to

$$\int^H \frac{z\,dz}{z^2 + az + b} = \log\left(\frac{h}{h_0}\right), \qquad h_0 = \text{const.} \tag{27}$$

Further, for $\Delta \equiv a^2 - 4b$, we must consider three cases separately.

2b.1. $\Delta = 0$. Equation (27) is integrated to

$$\left(h' + \frac{a}{2}\right) e^{a/(2h'+a)} = \frac{h}{h_0},$$

and h therefore satisfies the equation

$$H \equiv h' = f_1(h).$$

2b.2. $\Delta > 0 \Longrightarrow \Delta = c^2$. In this case, (27) is integrated to

$$\left(h' + \frac{a+c}{2}\right)^{c+a} \left(h' + \frac{a-c}{2}\right)^{c-a} = \left(\frac{h}{h_0}\right)^{2c}.$$

We rewrite it implicitly as

$$h' = f_2(h).$$

2b.3. $\Delta < 0 \Longrightarrow \Delta = -c^2$. In this case, (27) is integrated to

$$\left[1 + \left(\frac{2h' + a}{c}\right)^2\right] \exp\left[-2\frac{a}{c} \arctan\left(\frac{2h' + a}{c}\right)\right] = \left(\frac{h}{h_0}\right)^2.$$

We rewrite it implicitly as

$$h' = f_3(h).$$

In each case, h can be determined implicitly by

$$\int^h \frac{dz}{f_i(z)} = u,$$

B is given by

$$B_i(u) = -\frac{f_i^2 + af_i + b}{h^2}, \quad i = 1, 2, 3,$$

and A satisfies

$$h^2 A'' - h(f_i + a)A' - (2f_i^2 + 2af_i + 3b)A = 0.$$

The corresponding equation has a separable solution of the form

$$\int^u \frac{dz}{h(z)} = \phi(t) + \psi(x).$$

Conclusions

We have used the GCS approach to obtain solutions that are functionally separable. We note that these separable solutions cannot be obtained using the Lie classical symmetry method. The approach can be further developed to study higher nonlinear wave equations.

As can be easily seen, the explicit behavior of the solutions depends on the choice of many arbitrary constants. This means that there are many different cases. The explicit study of these cases and their physical and mathematical significance is beyond the scope of this paper and will be the subject of further research.

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