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Travelling wave solutions of two-dimensional Korteweg–de Vries–Burgers and Kadomtsev–Petviashvili equations

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Abstract

The travelling wave solutions of the two-dimensional Korteweg–de Vries–Burgers and Kadomtsev–Petviashvili equations are studied from two complementary points of view. The first one is an adaptation of the factorization technique that provides particular as well as general solutions. The second one applies the Painlevé analysis to both equations, throwing light on some aspects of the first method and giving an explanation to some restriction on the coefficients, as well as the relation between factorizations and integrals of motion.

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1. Introduction

The two-dimensional Korteweg–de Vries–Burgers (2D KdVB) equation is an extension of the KdVB equation, which is well known as a nonlinear model in the theory of plasmas and hydrodynamics. However, the 2D KdVB equation is not integrable, as it was the case of the KdVB equation that possesses conditionally the Painlevé property [1]. Therefore, it is important to develop specific methods to find exact solutions. Let us mention here some of them: the application of a special solution of square Hopf–Cole type to an ordinary differential equation [2], a computer algebra system (by using Mathematica) [3, 4], the tanh method [5, 6] and the first integral method [7]. On the other hand, the Kadomtsev–Petviashvili (KP) equation [8] is a natural two-dimensional generalization of the Korteweg–de Vries (KdV) equation. From the mathematical point of view this integrable equation has a rich structure, which has been considered in some textbooks [9, 10].

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In this work we will follow two approaches to obtain travelling wave solutions of both, 2D KdVB and KP equations. The first method consists of an application of the factorization technique well known in quantum mechanics. The second method starts by examining the Painlevé property of the reduced ordinary differential equation (ODE) and then changing it into a standard form by means of scale transformations. Of course, the final expressions for the travelling waves coincide, but the latter method supplies additional information: it gives an interpretation of factorizations in terms of a kind of invariants related to Bohlin's integral, and also explains some values of the parameters used in such factorizations. This second approach also allows us to find the Hamiltonian and Lagrangian functions associated with these nonlinear equations.

The paper is organized as follows. In section 2, we will briefly introduce the factorization technique adapted to nonlinear equations and show how to apply it to find the travelling wave solutions of the two-dimensional KdVB equation. In section 3, we analyse the KP equation by the same technique, obtaining a class of travelling wave solutions involving the Weierstrass function. Section 4 contains an analysis of the Painlevé property for the reduced 2D-KdVB ODE and the solutions through scale transformations. The specific values of the parameters, together with the factorization of a θ-dependent first integral, connect these results with those of section 2. In section 5 we consider the KP equation from this second point of view and establish the relation between these results and the factorizations of section 3. Finally, section 6 will end the paper with some conclusions and remarks.

2. Travelling waves of the two-dimensional KdVB equation

2.1. Travelling waves

The two-dimensional Korteweg–de Vries–Burgers equation has the following form [7]:

$$(u_t + \alpha u u_x + \beta u_{xx} + s u_{xxx})_x + \gamma u_{yy} = 0, \tag{2.1}$$

where α, β, s and γ are real constants which take into account different effects, such as nonlinearity, viscosity, turbulence, dispersion or dissipation. We remark that equation (2.1) is closely related to the Korteweg–de Vries–Burgers equation [1, 11]:

$$u_t + \alpha u u_x + \beta u_{xx} + s u_{xxx} = 0. \tag{2.2}$$

Let us assume that equation (2.1) has an exact solution in the form of a travelling wave

$$u(x, y, t) = \phi(\xi), \qquad \xi = hx + ly - \omega t, \tag{2.3}$$

where h, l, ω are real constants to be determined. If we substitute (2.3) in equation (2.1), we get

$$h^4 s\phi_{\xi\xi\xi\xi} + h^3\beta\phi_{\xi\xi\xi} + h^2\alpha(\phi\phi_\xi)_\xi + (\gamma l^2 - \omega h)\phi_{\xi\xi} = 0, \tag{2.4}$$

and then if we integrate equation (2.4) two times with respect to ξ, we have

$$h^4 s\phi_{\xi\xi} + h^3\beta\phi_\xi + \frac{h^2\alpha}{2}\phi^2 + (\gamma l^2 - \omega h)\phi = R_1\xi + R_2, \tag{2.5}$$

where R_1 and R_2 are two integration constants. The linear transformation of the dependent and independent variables

$$\xi = -hs\theta \qquad \phi(\xi) = -\frac{2}{\alpha s}W(\theta) \tag{2.6}$$

transforms (2.5) into

$$\frac{\mathrm{d}^2 W}{\mathrm{d}\theta^2} - \beta\frac{\mathrm{d}W}{\mathrm{d}\theta} - W^2 + kW = d_1\theta + d_2, \tag{2.7}$$

where

$$k = \frac{(\gamma l^2 - \omega h)s}{h^2}, \qquad d_1 = \frac{\alpha R_1 s^3}{2h}, \qquad d_2 = -\frac{\alpha R_2 s^2}{2h^2}. \tag{2.8}$$

We will assume in this section that $\beta \neq 0$. The case $\beta = 0$ corresponds to the KP equation and it will be separately studied in section 3.

2.2. Factorization of nonlinear equations

To deal with equation (2.9) we will introduce in this section a factorization technique applied to a class of nonlinear equations. So, let us consider the nonlinear second-order ODE

$$\frac{d^2U}{d\theta^2} - \beta\frac{dU}{d\theta} + F(U) = 0, \tag{2.9}$$

where $F(U)$ is a polynomial function. This equation can be factorized as

$$\left[\frac{d}{d\theta} - f_2(U,\theta)\right]\left[\frac{d}{d\theta} - f_1(U,\theta)\right]U(\theta) = 0, \tag{2.10}$$

with f_1 and f_2 being two unknown functions that may depend explicitly on U and θ. In order to find f_1 and f_2, we expand (2.10):

$$\frac{d^2U}{d\theta^2} - \left(f_1 + f_2 + \frac{\partial f_1}{\partial U}U\right)\frac{dU}{d\theta} + f_1 f_2 U - U\frac{\partial f_1}{\partial \theta} = 0, \tag{2.11}$$

and then comparing with (2.9) we obtain the following consistency conditions:

$$f_1 f_2 = \frac{F}{U} + \frac{\partial f_1}{\partial \theta} \tag{2.12}$$

$$f_2 + \frac{\partial(U f_1)}{\partial U} = \beta. \tag{2.13}$$

If we find a solution for this factorization problem, it will allow us to write a compatible first-order ODE

$$\left[\frac{d}{d\theta} - f_1(U,\theta)\right]U(\theta) = 0 \tag{2.14}$$

that provides a (particular) solution to the nonlinear equation (2.9) (see [11]).

2.3. Factorization of the KdVB travelling wave equation

If we want to use the factorization technique described above in order to solve (2.7), the first integration constant R_1 must be taken equal to zero. Also, it can be shown that nontrivial factorizations are obtained only when $d_2 = 0$, which is a very restrictive condition. To circumvent this constraint, we propose a simple displacement on the unknown function

$$W(\theta) = U(\theta) + \delta, \tag{2.15}$$

where δ is a constant solution of (2.7) so that

$$d_2 = k\delta - \delta^2. \tag{2.16}$$

We will restrict to values of d_2 such that $k^2 > 4d_2$ in order to have real solutions of (2.16). In particular we select the solution of (2.16) such that $2\delta = k - \sqrt{k^2 - 4d_2}$ and therefore $k - 2\delta > 0$.

According to this change, (2.7) becomes

$$\frac{d^2U}{d\theta^2} - \beta\frac{dU}{d\theta} - U^2 + (k - 2\delta)U = 0. \tag{2.17}$$

If we compare equations (2.17) and (2.11), condition (2.12) adopts the form

$$f_1 f_2 = k - 2\delta - U + \frac{\partial f_1}{\partial\theta}. \tag{2.18}$$

2.3.1. Particular case. We get particular solutions of (2.17) in the case in which f_1 and f_2 do not depend explicitly on θ. By using the ansatz

$$f_1 = AU^p + B \tag{2.19}$$

we get from (2.18)

$$f_2 = \beta - B - A(p+1)U^p, \qquad p = \frac{1}{2}, \qquad A^2 = \frac{2}{3}, \qquad B = \frac{2\beta}{5} \tag{2.20}$$

together with a constraint between the parameters k, δ and β

$$k - 2\delta = \frac{6\beta^2}{25} \quad\Rightarrow\quad d_2 = \frac{k^2}{4} - \frac{9\beta^4}{625}. \tag{2.21}$$

The particular solutions of (2.17) are obtained by solving (2.14). Indeed, we get the following two ODEs:

$$\frac{dU^\pm}{d\theta} = \pm\sqrt{\frac{2}{3}}U^{3/2} + \frac{2\beta}{5}U, \tag{2.22}$$

whose general solutions are

$$U^\pm(\theta) = \frac{6\beta^2}{25}\frac{1}{\left(1 \pm e^{-\frac{\beta(\theta-\theta_0)}{5}}\right)^2}, \tag{2.23}$$

where θ_0 is an integration constant. These solutions can be written in the following form:

$$U^+(\theta) = \frac{3\beta^2}{50}\left(1 + \tanh\left[\frac{\beta(\theta-\theta_0)}{10}\right]\right)^2 \tag{2.24}$$

$$U^-(\theta) = \frac{3\beta^2}{50}\left(1 + \coth\left[\frac{\beta(\theta-\theta_0)}{10}\right]\right)^2. \tag{2.25}$$

Now, according to (2.3), (2.6) and (2.15) we can write the solutions of the 2D KdVB equation as

$$u(x, y, t) = -\frac{2}{\alpha s}(U(\theta) + \delta). \tag{2.26}$$

Substituting δ and $U(\theta)$ into (2.26), we obtain the particular travelling solitary wave solutions:

$$u(x, y, t) = -\frac{12\beta^2}{25\alpha s}\frac{1}{\left(1 + z_0\, e^{\frac{\beta}{5sh}(hx+ly-\omega t)}\right)^2} + \frac{\omega h - \gamma l^2}{\alpha h^2} + \frac{6\beta^2}{25\alpha s}, \tag{2.27}$$

where $z_0 = \pm e^{\frac{\beta}{5}\theta_0}$ is an arbitrary constant. This solution coincides with the result of [7].

2.3.2. General case. We can also find the general solution to the factorization of equation (2.17) with f_1 and f_2 depending only on U. To do this, first let us substitute (2.13) into equation (2.18),

$$Uf_1\frac{\mathrm{d}f_1}{\mathrm{d}U} = \beta f_1 - f_1^2 + U - (k-2\delta), \tag{2.28}$$

and then let us make the following replacement:

$$g(U) = f_1(U)U, \tag{2.29}$$

then (2.28) adopts the form of an Abel equation of the second kind:

$$g\frac{\mathrm{d}g}{\mathrm{d}U} - \beta g = U^2 - (k-2\delta)U. \tag{2.30}$$

Provided that restriction (2.21) between the parameters is satisfied, the solution of (2.30) is given in the parametric form [13] as

$$U(\tau) = \frac{6\beta^2}{25}\tau^2\wp(\tau), \qquad g(\tau) = \frac{6\beta^3}{125}\tau^2 E_4(\tau), \tag{2.31}$$

where

$$E_4(\tau) = \tau\sqrt{4\wp^3(\tau)-1} + 2\wp(\tau), \tag{2.32}$$

and $\wp(\tau) \equiv \wp(\tau + C_2, 0, 1)$ is a particular case of the elliptic Weierstrass function including an integration constant C_2.

If we derive $U(\tau)$, given in (2.31), with respect to τ, replacing $\wp'(\tau) = \sqrt{4\wp(\tau)^3 - 1}$ and $g(\tau)$, we get

$$\frac{\beta\tau}{5}\frac{\mathrm{d}U}{\mathrm{d}\tau} - f_1(U)U = 0, \tag{2.33}$$

then, comparing with equation (2.16), we obtain τ as a function of θ, i.e., $\tau = \mathrm{e}^{\beta(\theta+\theta_0)/5}$, where θ_0 is an integration constant. If we use this value of τ in (2.31) we get

$$U(\theta) = \frac{6\beta^2}{25}\,\mathrm{e}^{2\beta(\theta+\theta_0)/5}\wp\left(\mathrm{e}^{\beta(\theta+\theta_0)/5} + C_2, 0, 1\right). \tag{2.34}$$

Now, we can write the exact solution of equation (2.1) using (2.26) and substituting δ and θ,

$$u(x,y,t) = -\frac{12\beta^2}{25\alpha s}\,\mathrm{e}^{\frac{-2\beta}{5sh}\xi}\,\mathrm{e}^{\frac{2\theta_0}{5}}\,\wp\left(\mathrm{e}^{\frac{-\beta}{5sh}\xi}\mathrm{e}^{\frac{\theta_0}{5}} + C_2, 0, 1\right) + \frac{\omega h - \gamma l^2}{\alpha h^2} + \frac{6\beta^2}{25\alpha s}, \tag{2.35}$$

with ξ given in (2.3). This solution can be written in terms of Jacobian elliptic functions as

$$u(x,y,t) = -\frac{\sqrt{3}\beta^2}{25\alpha s}\,\mathrm{e}^{-\frac{2\beta}{5hs}\xi}a_0^2\left(1+\sqrt{3}\frac{1+\mathrm{cn}\left[a_0\left(\mathrm{e}^{-\frac{\beta}{5hs}\xi}+z_0\right), m\right]}{1-\mathrm{cn}\left[a_0\left(\mathrm{e}^{-\frac{\beta}{5hs}\xi}+z_0\right), m\right]}\right) + \frac{\omega h - \gamma l^2}{\alpha h^2} + \frac{6\beta^2}{25\alpha s}, \tag{2.36}$$

where $m = 1/2 - \sqrt{3}/4$, $a_0 = 2\,\mathrm{e}^{\beta\theta_0/5}$ and $z_0 = C_2\,\mathrm{e}^{-\beta\theta_0/5}$. Solution (2.35) also coincides with the result of [7], obtained in another way.

Let us remark that in the limit $a_0 \to 0$, the elliptic function $\mathrm{cn}\left[a_0\left(\mathrm{e}^{-\frac{\beta}{5hs}\xi}+z_0\right), m\right]$ can be expanded in Taylor series as $1 - \frac{1}{2}a_0^2\left(\mathrm{e}^{-\frac{\beta}{5hs}\xi}+z_0\right)^2 + \cdots$. Therefore, in the limit $a_0 \to 0$, (2.36) reduces to (2.27). If we take $\gamma = 0$ in solutions (2.27) and (2.35), we get the travelling waves of the KdVB equation (2.2) given in [11].

Note that from equations (2.31)–(2.33) we can write the expression for $f_1(U)$:

$$f_1(U) = \frac{1}{U}\left[\frac{2}{3}\left(U^3 - C_0^3\,\mathrm{e}^{\frac{6\beta}{5}\theta(U)}\right)\right]^{1/2} + \frac{2\beta}{5}, \tag{2.37}$$

where C_0 is an arbitrary constant and θ is an implicit function of U given in (2.31). The function $f_2(U)$ can also be found by (2.13) but taking into account the dependence of θ on U

$$f_2(U) = -\frac{U^2 - \frac{2\beta}{5}C_0^3\,\mathrm{e}^{\frac{6\beta}{5}\theta(U)}\frac{\mathrm{d}\theta}{\mathrm{d}U}}{\left[\frac{2}{3}\left(U^3 - C_0^3\,\mathrm{e}^{\frac{6\beta}{5}\theta(U)}\right)\right]^{1/2}} + \frac{3\beta}{5}, \qquad \frac{\mathrm{d}\theta}{\mathrm{d}U} = \frac{1}{Uf_1(U)}. \tag{2.38}$$

Then after a straightforward calculation the consistency equation (2.12) with $\frac{\partial f_1}{\partial\theta} = 0$ is satisfied by $f_1(U)$ and $f_2(U)$ having in mind restriction (2.21).

Due to the parametric relation (2.31) between U and θ, expression (2.37) of $f_1(U)$ in terms of U cannot be displayed explicitly. Hence, we can define a new $f_1(U,\theta)$ as an explicit expression on both variables U and θ in the form

$$f_1(U,\theta) = \frac{1}{U}\left[\frac{2}{3}\left(U^3 - C_0^3\,\mathrm{e}^{\frac{6\beta}{5}\theta}\right)\right]^{1/2} + \frac{2\beta}{5} \tag{2.39}$$

and therefore

$$f_2(U,\theta) = -\frac{U^2}{\left[\frac{2}{3}\left(U^3 - C_0^3\,\mathrm{e}^{\frac{6\beta}{5}\theta}\right)\right]^{1/2}} + \frac{3\beta}{5}. \tag{2.40}$$

It is worth to remark that these expressions constitute a generalization of the ansatz (2.19)–(2.20) used to get the particular solutions (2.27). In this case we have to use the consistency conditions for the factorization given in (2.12)–(2.13) with $\frac{\partial f_1}{\partial\theta} \neq 0$. Then, we can check that indeed these conditions are satisfied by $f_1(U,\theta)$, $f_2(U,\theta)$ given in (2.39) and (2.40). Therefore, equation (2.14) takes the form

$$\frac{\mathrm{d}U}{\mathrm{d}\theta} = \left[\frac{2}{3}\left(U^3 - C_0^3\,\mathrm{e}^{\frac{6\beta}{5}\theta}\right)\right]^{1/2} + \frac{2\beta}{5}U. \tag{2.41}$$

Of course, (2.41) has the same solution as given above in (2.34).

3. Travelling waves of the two-dimensional KP equation

By taking $\beta = 0$ in (2.1), we obtain the two-dimensional Kadomtsev–Petviashvili (KP) equation:

$$(u_t + \alpha u u_x + s u_{xxx})_x + \gamma u_{yy} = 0, \tag{3.1}$$

where α, s and γ are real constants different from zero. Sometimes this equation is introduced in different standard forms by rescaling the coordinates. For instance in [10, 14], it is written as

$$(-4u_t + 6uu_x + u_{xxx})_x + 3\sigma^2 u_{yy} = 0. \tag{3.2}$$

When $\sigma^2 = -1$ it is referred to as KPI and if $\sigma^2 = 1$, as KPII. These two types have different properties from both physical and mathematical points of view. Here we will consider the KP equation with general coefficients.

Suppose that equation (3.1) has travelling wave solution. Operating as in section 2, we make the changes,

$$u(x,y,t) = -\frac{2}{\alpha s}(U(\theta) + \delta), \qquad \theta = -\frac{hx + ly - \omega t}{hs} \tag{3.3}$$

that transform (3.1) into the equation

$$\frac{\mathrm{d}^2U}{\mathrm{d}\theta^2} - U^2 + (k - 2\delta)U = 0, \tag{3.4}$$

where k and δ are defined in (2.8) and (2.16), but in this case there will be no restrictions on these parameters. To apply the factorization technique we will assume that f_1 and f_2 depend only on U. Thus, comparing (3.4) with (2.11) we have

$$f_1 f_2 = k - 2\delta - U \tag{3.5}$$

$$f_2 + \frac{\partial(U f_1)}{\partial U} = 0. \tag{3.6}$$

Replacing (3.6) in equation (3.5) we get a nonlinear ODE for f_1,

$$f_1 \frac{\mathrm{d}f_1}{\mathrm{d}U} = 1 - (k - 2\delta)U^{-1} - f_1^2 U^{-1}. \tag{3.7}$$

The general solutions of this equation are of two types:

$$f_1 = \pm\sqrt{\frac{2}{3}U - (k - 2\delta) + cU^{-2}}, \tag{3.8}$$

where c is an integration constant. For this case, equation (2.14) becomes

$$\frac{\mathrm{d}U}{\sqrt{\frac{2}{3}U^3 - (k - 2\delta)U^2 + c}} = \pm\mathrm{d}\theta. \tag{3.9}$$

The solution of (3.9) is

$$U(\theta) = 6\wp((\theta + \theta_1), g_2, g_3) + \frac{(k - 2\delta)}{2}, \tag{3.10}$$

where θ_1 is an integration constant, and

$$g_2 = \frac{(k - 2\delta)^2}{12}, \qquad g_3 = -\frac{(k - 2\delta)^3}{216} - \frac{c}{36}. \tag{3.11}$$

Therefore, we can write the general travelling wave solution of the KP equation using (3.3) as

$$u(x, y, t) = -\frac{12}{\alpha s}\wp\left(\frac{(hx + ly - \omega t) + \theta_1}{\alpha s}, g_2, g_3\right) + \frac{\omega h - l^2\gamma}{\alpha h^2}. \tag{3.12}$$

The general solution (3.10) can also be written in terms of Jacobi elliptic functions as follows:

$$U(\theta) = \frac{k - 2\delta}{2} - 2a_0^2(m^2 + 1) + 6a_0^2 m^2 \mathrm{sn}^2\,[a_0(\theta - \theta_0), m]\,, \tag{3.13}$$

where θ_0 is an integration constant and a_0, m are positive and satisfy

$$k - 2\delta = 4a_0^2\sqrt{m^4 - m^2 + 1} \tag{3.14}$$

$$c = \frac{(k - 2\delta)^3}{6} - \frac{16a_0^6}{3}(m^2 + 1)(2m^2 - 1)(m^2 - 2). \tag{3.15}$$

3.1. Hyperbolic limits

If we choose $c = (k - 2\delta)^3/3$ in (3.11) we have $g_2 = 12\mu^2$ and $g_3 = -8\mu^3$, with $\mu = (k - 2\delta)/12$, we get the particular solution of the KP equation:

$$u(x, y, t) = -\frac{3r}{\alpha h^2}\mathrm{cosech}^2\left(\frac{r^{1/2}}{2s^{1/2}h^2}(hx + ly - \omega t + \theta_1)\right) + \frac{\omega h - l^2\gamma - r}{\alpha h^2}, \tag{3.16}$$

where $r = \sqrt{2\alpha R_2 h^2 + (l^2\gamma - \omega h)^2}$. By setting $\theta_1 = \frac{\pi}{2}\mathrm{i} + \theta_0$, solution (3.16) becomes

$$u(x, y, t) = \frac{3r}{\alpha h^2}\mathrm{sech}^2\left(\frac{r^{1/2}}{2s^{1/2}h^2}(hx + ly - \omega t + \theta_0)\right) + \frac{\omega h - l^2\gamma - r}{\alpha h^2}. \tag{3.17}$$

If we take $r = h^4 s$ and $\alpha = 1$, these solutions coincide with the result of [3], provided that the dispersion relation $\omega h - l^2\gamma - h^4 s = 0$ is satisfied. These two particular solutions could be obtained from the hyperbolic limits of the Jacobi expression (3.13) by taking the value of the elliptic parameter $m = 1$.

When we choose $\gamma = 0$ in equation (3.1), it becomes the KdV equation. Hence, the travelling wave solutions of the KdV equation are also given by (3.12), and the particular solutions by (3.16), (3.17) with the replacement $\gamma = 0$.

4. The Painlevé analysis

As we have seen in the previous sections, the factorization technique is only applicable for some values of the coefficients. In order to have a better understanding of the links existing between the coefficients, let us apply the Painlevé test [15] to equation (2.17). This test requires that all the solutions of (2.17) are single valued in the neighbourhood of a movable singularity depending on the initial conditions. From a technical point of view, it means that the solutions must be expressed locally as a Laurent series of the form

$$U(\theta) = \sum_{j=0}^{\infty} a_j(\theta - \theta_0)^{j-2} \qquad a_0 \neq 0 \tag{4.1}$$

in a neighbourhood of θ_0. The substitution of (4.1) in (2.17) provides the recursion relation

$$\begin{aligned}[(j-2)(j-3) - 2a_0]\, a_j &= (k - 2\delta)a_{j-2} + \beta(j-3)a_{j-1} \\ &\quad + \sum_{m=1}^{j-1} a_m a_{j-m}, \qquad j = 0, 1, \ldots, \end{aligned} \tag{4.2}$$

where $a_{-1} = a_{-2} = 0$. For $j = 0$, equation (4.2) provides the leading term $a_0 = 6$. Equation (4.2) allows us to obtain every a_j in terms of a_m, with $m \in \{0, 1, \ldots, j-1\}$, except when its coefficient $[(j-2)(j-3) - 2a_0]$ vanishes, that is for $j = 6$. Therefore a_6 remains arbitrary and it is called a resonance of the equation. For $j = 6$, (4.2) is the resonance condition that must be satisfied by $a_1, \ldots, a_5$ in order to pass the Painlevé test. Taking $j = 1, \ldots, 5$ in (4.2), we have

$$\begin{aligned} a_1 &= \frac{6\beta}{5} \qquad a_2 = \frac{k}{2} - \delta - \frac{\beta^2}{50} \qquad a_3 = \frac{\beta^3}{250} \\ a_4 &= \frac{k^2}{40} - \frac{k\delta}{10} + \frac{\delta^2}{10} - \frac{\beta^4}{5000} \\ a_5 &= -\frac{\beta k^2}{600} + \frac{11\beta k\delta}{150} - \frac{11\beta\delta^2}{150} + \frac{79\beta^5}{75\,000}. \end{aligned} \tag{4.3}$$

For $j = 6$, (4.2) provides the following resonance condition:

$$-3\beta a_5 - 2a_1 a_5 - 2a_2 a_4 - a_3^2 + (k - 2\delta)a_4 = 0. \tag{4.4}$$

By substituting (4.3) in the previous equation (4.4), we have

$$\beta^2\left[\beta^4 - \left(\tfrac{25}{6}(k - 2\delta)\right)^2\right] = 0. \tag{4.5}$$

Therefore, (2.17) passes the Painlevé test in two cases:

- (a) $\beta^2 = 25(k - 2\delta)/6$ that corresponds to the case in which the KdVB equation factorizes according to (2.21). This means that reduction (2.17) of the 2D KdVB equation is integrable only under this restriction.
- (b) $\beta = 0$ that corresponds to the KP equation that passes the Painlevé test for all the values of the coefficients [16].

5. Scale transformations

Now let us study (2.17) from a different point of view. In the previous section we have established condition (4.5) to be satisfied in order to pass the Painlevé test. Therefore, under this assumption equation (2.17) should be written in a canonical form as one of the 50 integrable second-order ODEs classified by Painlevé and co-workers [17].

5.1. KdVB case

For $\beta \neq 0$, the change [17]

$$U(\theta) = \lambda^2(\theta) Y(z), \qquad \mathrm{d}z = \lambda(\theta)\,\mathrm{d}\theta, \tag{5.1}$$

where

$$\lambda(\theta) = \frac{\beta}{5}\,\mathrm{e}^{\frac{\beta}{5}\theta} \quad \Longrightarrow \quad z = \mathrm{e}^{\frac{\beta}{5}\theta} \tag{5.2}$$

allows us to write (2.17) in the canonical form

$$\frac{\mathrm{d}^2 Y}{\mathrm{d}z^2} - Y^2 = 0, \tag{5.3}$$

if and only if condition (2.21) is satisfied.

5.1.1. θ-dependent first integral. Equation (5.3) can be trivially integrated as

$$E = \frac{1}{2}\left(\frac{\mathrm{d}Y}{\mathrm{d}z}\right)^2 - \frac{Y^3}{3}, \tag{5.4}$$

where E is a constant that can be written in the initial variables as the following 'θ-dependent first integral' for (2.17):

$$E = \left(\frac{5}{\beta}\right)^6 \mathrm{e}^{-\frac{6\beta}{5}\theta}\left(\frac{1}{2}\left(\frac{\mathrm{d}U}{\mathrm{d}\theta} - \frac{2}{5}\beta U\right)^2 - \frac{U^3}{3}\right). \tag{5.5}$$

5.1.2. Factorization. Equation (5.4) factorizes trivially as the product

$$\left(\frac{\mathrm{d}Y}{\mathrm{d}z} - \sqrt{\frac{2}{3}Y^3 + 2E}\right)\left(\frac{\mathrm{d}Y}{\mathrm{d}z} + \sqrt{\frac{2}{3}Y^3 + 2E}\right) = 0, \tag{5.6}$$

therefore

$$\frac{\mathrm{d}Y}{\mathrm{d}z} \pm \sqrt{\frac{2}{3}Y^3 + 2E} = 0 \tag{5.7}$$

solves (5.3). If we write (5.7) in terms of the original variables, we have

$$\frac{\mathrm{d}U}{d\theta} - \frac{2\beta}{5}U \pm \sqrt{\frac{2}{3}U^3 + 2E\left(\frac{\beta}{5}\right)^6 \mathrm{e}^{\frac{6\beta}{5}\theta}} = 0 \tag{5.8}$$

that coincides with factorization (2.41) of section 2 by identifying $3E = -(25C_0/\beta^2)^3$ and the particular case (2.22) when $E = 0$.

5.1.3. Bohlin's integral. As it is well known [18], for the damped harmonic oscillator, it is possible to define two independent integrals of motion related to Bohlin's integral such that the 'time-dependent energy' of the damped harmonic oscillator factorizes as the product of these invariants. In the present case, the form of (5.4) suggests the following factorization:

$$E = \frac{1}{2}D_1 D_2 \qquad D_1 = \left(\frac{\mathrm{d}Y}{\mathrm{d}z} - \sqrt{\frac{2}{3}Y^3}\right)\mathrm{e}^{M} \qquad D_2 = \left(\frac{\mathrm{d}Y}{\mathrm{d}z} + \sqrt{\frac{2}{3}Y^3}\right)\mathrm{e}^{-M}. \tag{5.9}$$

It is easy to check that $\frac{\mathrm{d}I_j}{\mathrm{d}z} = 0$, $j = 1, 2$ if M is defined as

$$M = \int \sqrt{\frac{3}{2}Y}\,\mathrm{d}z. \tag{5.10}$$

If we write the invariants in the original variables, we have

$$\begin{aligned} D_1 &= \left(\frac{5}{\beta}\right)^3 \mathrm{e}^{-\frac{3\beta}{5}\theta}\left(\frac{\mathrm{d}U}{\mathrm{d}\theta} - \frac{2\beta}{5}U - \sqrt{\frac{2}{3}U^3}\right)\mathrm{e}^{\int\sqrt{3U/2}\,\mathrm{d}\theta} \\ D_2 &= \left(\frac{5}{\beta}\right)^3 \mathrm{e}^{-\frac{3\beta}{5}\theta}\left(\frac{\mathrm{d}U}{\mathrm{d}\theta} - \frac{2\beta}{5}U + \sqrt{\frac{2}{3}U^3}\right)\mathrm{e}^{-\int\sqrt{3U/2}\,\mathrm{d}\theta}. \end{aligned} \tag{5.11}$$

Therefore, D_1 and D_2 are constant of motion for (2.17) if relation (2.21) holds.

5.1.4. Hamiltonian Formalism. We can obtain a Lagrangian and a Hamiltonian for (2.17), starting from the canonical equation (5.3) that obviously can be derived from the Lagrangian

$$L(Y, Y_z) = \tfrac{1}{2}Y_z^2 + \tfrac{1}{3}Y^3. \tag{5.12}$$

The momentum is

$$P = \frac{\partial L}{\partial Y_z} = Y_z, \tag{5.13}$$

and therefore the Hamiltonian $H = Y_z P - L$ is

$$H(Y, P) = \tfrac{1}{2}P^2 - \tfrac{1}{3}Y^3. \tag{5.14}$$

In order to derive a Lagrangian $\widetilde{L}(U, U_\theta)$ for (2.17), it is necessary to consider that the action principle should be invariant under the transformation (5.1). Therefore,

$$L(Y, Y_z)\,\mathrm{d}z = \widetilde{L}(U, U_\theta)\,\mathrm{d}\theta \quad\Rightarrow\quad \widetilde{L}(U, U_\theta) = \lambda(\theta)L(Y, Y_z). \tag{5.15}$$

Combining (5.1) and (5.12) we have

$$\widetilde{L}(U, U_\theta) = \frac{1}{\lambda(\theta)^5}\left[\frac{1}{2}\left(U_\theta - \frac{2\beta}{5}U\right)^2 + \frac{U^3}{3}\right] \tag{5.16}$$

and

$$\widetilde{P} = \frac{\partial\widetilde{L}}{\partial U_\theta} = \frac{1}{\lambda(\theta)^5}\left(U_\theta - \frac{2\beta}{5}U\right). \tag{5.17}$$

Then, we have for (2.17) the non-autonomous Hamiltonian $\widetilde{H} = U_\theta\widetilde{P} - \widetilde{L}(U, U_\theta)$,

$$\widetilde{H} = \frac{1}{2}\lambda(\theta)^5\widetilde{P}^2 + \frac{2\beta}{5}\widetilde{P}U - \frac{1}{3\lambda(\theta)^5}U^3, \tag{5.18}$$

that obviously is not a constant of motion. Nevertheless, the θ-dependent constant of motion E given in (5.5) can be expressed in the phase variables U, $\widetilde{P}$ as

$$E = \frac{1}{2}\lambda(\theta)^4\widetilde{P}^2 - \frac{1}{3}\frac{U^3}{\lambda(\theta)^6}. \tag{5.19}$$

It is easy to check that (5.19) satisfies

$$\frac{\mathrm{d}E}{\mathrm{d}\theta} = \frac{\partial E}{\partial\theta} + \{E, \widetilde{H}\} = 0. \tag{5.20}$$

5.2. KP case

Equation (3.4) is directly written in the canonical form and it can be trivially integrated as

$$\frac{1}{2}\left(\frac{\mathrm{d}U}{\mathrm{d}\theta}\right)^2 - \frac{U^3}{3} + \frac{k-2\delta}{2}U^2 = E, \tag{5.21}$$

where E is a constant of integration.

5.2.1. Factorization. From (5.21), we have

$$\left(\frac{\mathrm{d}U}{\mathrm{d}\theta} - \sqrt{\frac{2}{3}U^3 - (k-2\delta)U^2 + 2E}\right)\left(\frac{\mathrm{d}U}{\mathrm{d}\theta} + \sqrt{\frac{2}{3}U^3 - (k-2\delta)U^2 + 2E}\right) = 0 \tag{5.22}$$

that coincides with factorization (3.9) if $c = 2E$.

5.2.2. Integrals of motion. A different form of (5.21) is

$$\left(\frac{\mathrm{d}U}{d\theta} + \sqrt{\frac{2}{3}U^3 - (k-2\delta)U^2}\right)\left(\frac{\mathrm{d}U}{\mathrm{d}\theta} - \sqrt{\frac{2}{3}U^3 - (k-2\delta)U^2}\right) = 2E \tag{5.23}$$

that suggests [18] that E can be written as the product of two first integrals defined as

$$D_1 = \left(\frac{\mathrm{d}U}{d\theta} - \sqrt{\frac{2}{3}U^3 - (k-2\delta)U^2}\right)\mathrm{e}^{N} \tag{5.24}$$

$$D_2 = \left(\frac{\mathrm{d}U}{d\theta} + \sqrt{\frac{2}{3}U^3 - (k-2\delta)U^2}\right)\mathrm{e}^{-N}. \tag{5.25}$$

It is easy to prove that D_1, D_2 are constants of motion if

$$N = \int \frac{(U^2 - (k-2\delta)U}{\sqrt{\frac{2}{3}U^3 - (k-2\delta)U^2}}\,\mathrm{d}\theta. \tag{5.26}$$

Here, the Lagrangian and the Hamiltonian functions can also be straightforwardly obtained, as it was done in the KdVB case.

6. Conclusions

In this paper we have investigated the travelling wave solutions of the two-dimensional KdVB and KP equations. We started by applying a factorization technique, already used in other nonlinear equations, to get particular and general solutions in terms of elliptic functions. In a particular case (section 2.3.2), we have adapted the method so that the functions f_1 and f_2 entering the factorization may depend explicitly on both variables U and θ. The factorization can be realized only by imposing some restrictions on the coefficients of the KdVB equation. These constrains are shown to coincide with those obtained by means of the Painlevé analysis of this equation. Afterwards, by using scale transformations we obtained a kind of θ-dependent first integrals that are directly related to the factorizations. From these integrals we formulated a kind of Bohling invariants. Using this framework we also derived the Lagrangians and Hamiltonians for the nonlinear equations considered in this paper. Finally, let us stress that, compared with the methods of other references, our approach is quite systematic and gives a comprehensive picture for this kind of travelling solutions.

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