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RECIPROCAL TRANSFORMATIONS FOR A SPECTRAL PROBLEM IN 2+1 DIMENSIONS

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We present two reciprocal transformations for a spectral problem in 2+1 *dimensions. Reductions of the transformed equations to* 1+1 *dimensions include the Degasperis–Procesi and Vakhnenko–Parkes equations.*

Keywords: reciprocal transformation, spectral problem

1. Introduction

In [1]–[3], we considered the application of the singular manifold method to equations in 2+1 dimensions of the form

$$\left(H_{x_1x_1x_2}+3H_{x_2}H_{x_1}+n_0\frac{H_{x_1x_2}^2}{H_{x_2}}\right)_{x_1}=H_{x_2x_3}. \tag{1.1}$$

We studied the cases $n_0=0$ and $n_0=-3/4$ of (1.1) in [1] and [2] and derived their Lax pair using the singular manifold method [4]. Based on the results in those papers, we propose a spectral problem of the form

$$\phi_{x_1x_1x_1}-\phi_{x_3}+3H_{x_1}\phi_{x_1}+b_1H_{x_1x_1x_1}\phi=0,$$

$$\phi_{x_1x_2}+H_{x_2}\phi+b_2\frac{H_{x_1x_2}}{H_{x_2}}\phi_{x_2}=0.$$

It is easy to verify that the compatibility condition for this spectral problem implies the conditions on the coefficients

$$b_1+3b_2=0,\qquad (b_2+1)(2b_2+1)=0.$$

These conditions can be rewritten as

$$b_2=\frac{k-5}{6},\qquad b_1=-\frac{k-5}{2}, \tag{1.2}$$

where

$$(k+1)(k-2)=0\Longrightarrow k^2=k+2. \tag{1.3}$$

We note that these values of k correspond precisely to the cases analyzed in [1], [2]. Therefore, the spectral problem is

$$\begin{aligned}&\phi_{x_1x_1x_1}-\phi_{x_3}+3H_{x_1}\phi_{x_1}-\frac{k-5}{2}H_{x_1x_1}\phi=0,\\&\phi_{x_1x_2}+H_{x_2}\phi+\frac{k-5}{6}\frac{H_{x_1x_2}}{H_{x_2}}\phi_{x_2}=0,\end{aligned} \tag{1.4}$$

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and the (2+1)-dimensional equation arising from this spectral problem can be written as

$$H_{x_1x_1x_2} + 3H_{x_2}H_{x_1} - \frac{k+1}{4}\frac{H_{x_1x_2}^2}{H_{x_2}} = \Omega,$$
$$\Omega_{x_1} = H_{x_2x_3}. \tag{1.5}$$

As proved in [1], [2], the singular manifold method, which is based on the Painlevé property [4], is an excellent instrument for deriving the Lax pair of many equations. Unfortunately, the Painlevé property depends strongly on the variables in which the equation is written. Nevertheless, we can sometimes identify nontrivial transformations that transform an equation into a form in which the Painlevé methods work. For instance, in [5], we applied a reciprocal transformation [6]–[9] to a (2+1)-dimensional Camassa–Holm hierarchy that allowed transforming it into a system of equations with which the singular manifold method can be successfully used [10].

Our main aim here is to identify some reciprocal transformations for (1.5). We prove that the reciprocal transformations that we construct transform (1.5) to a system that generalizes the Vakhnenko [11] and/or Degasperis–Procesi [7] equations to 2+1 dimensions. The spectral problem for these equations can be easily derived from the reduction of the transformed (1.4) to 1+1 dimensions.

A connection between the Degasperis–Procesi and Vakhnenko–Parkes equations was shown in [12], [13]. We prove that both equations arise as 1+1 reductions of reciprocal transformations of (1.5).

2. First reciprocal transformation

We construct a reciprocal transformation of the form

$$dx_1 = \alpha(x,t,T)[dx - \beta(x,t,T)\,dt - \epsilon(x,t,T)\,dT],$$
$$x_2 = t, \qquad x_3 = T, \tag{2.1}$$

which means that

$$\frac{\partial}{\partial x_1} = \frac{1}{\alpha}\frac{\partial}{\partial x}, \qquad \frac{\partial}{\partial x_2} = \frac{\partial}{\partial t} + \beta\frac{\partial}{\partial x}, \qquad \frac{\partial}{\partial x_3} = \frac{\partial}{\partial T} + \epsilon\frac{\partial}{\partial x}, \tag{2.2}$$

and the cross derivatives of (2.1) obviously imply that

$$\alpha_t + (\alpha\beta)_x = 0, \qquad \alpha_T + (\alpha\epsilon)_x = 0, \qquad \beta_T - \epsilon_t + \epsilon\beta_x - \beta\epsilon_x = 0. \tag{2.3}$$

If we select the transformation in the form

$$H_{x_2}(x_1,x_2,x_3) = [\alpha(x,t,T)]^k, \tag{2.4}$$

where k satisfies (1.3), then we easily obtain

$$H_{x_1}(x_1,x_2,x_3) = \frac{1}{3}\left[\frac{\Omega}{\alpha^k} - k\frac{\alpha_{xx}}{\alpha^3} + (2k-1)\left(\frac{\alpha_x}{\alpha^2}\right)^2\right] \tag{2.5}$$

from (1.5) and

$$\Omega_x = -k\alpha^{(k+1)}\epsilon_x \tag{2.6}$$

from (1.5), (2.3), and (2.4). Furthermore, the compatibility condition $H_{x_2x_1} = H_{x_1x_2}$ between (2.4) and (2.5) yields

$$\Omega_t = -\beta\Omega_x - k\Omega\beta_x + \alpha^{k-2}\left[-k\beta_{xxx} + (k-2)\beta_{xx}\frac{\alpha_x}{\alpha} + 3k\alpha^k\alpha_x\right]. \tag{2.7}$$

Equations (2.3), (2.6), and (2.7) in fact constitute the transformation of system (1.5). Nevertheless, a more convenient form arises if we set

$$A_1 = \frac{k+1}{3}, \qquad A_2 = \frac{2-k}{3}, \tag{2.8}$$

$$M = \frac{1}{\alpha^3}. \tag{2.9}$$

Integrability condition (1.3) can be written as

$$A_1 A_2 = 0, \qquad A_1 + A_2 = 1. \tag{2.10}$$

Using these definitions, we write (2.3), (2.6), and (2.7) as the system

$$\begin{gathered}
A_1\left(\Omega_t + \beta\Omega_x + 2\Omega\beta_x + 2\beta_{xxx} + 2\frac{M_x}{M^2}\right)M + \\
+ A_2(\Omega_t + \beta\Omega_x - \Omega\beta_x - M\beta_{xxx} - M_x\beta_{xx} - M_x) = 0, \\
A_1\left(\Omega_x + 2\frac{\epsilon_x}{M}\right) + A_2(\Omega_x - \epsilon_x) = 0, \\
M_t = 3M\beta_x - \beta M_x, \qquad M_T = 3M\epsilon_x - \epsilon M_x, \qquad \beta_T - \epsilon_t + \epsilon\beta_x - \beta\epsilon_x = 0.
\end{gathered} \tag{2.11}$$

The reciprocal transformation can be also applied to spectral problem (1.4), and after some direct calculations, we obtain

$$\psi_{xt} = A_1\left[-\beta\psi_{xx} + \left(\beta_{xx} - \frac{1}{M}\right)\psi\right] + A_2[-\beta\psi_{xx} - 2\beta_x\psi_x - (1 + \beta_{xx})\psi], \tag{2.12}$$

$$\psi_T = A_1[M\psi_{xxx} + (M\Omega - \epsilon)\psi_x] + A_2[M\psi_{xxx} + 2M_x\psi_{xx} + (M_{xx} + \Omega - \epsilon)\psi_x], \tag{2.13}$$

where we set

$$\Phi(x_1, x_2, x_3) = \alpha^{(2k-1)/3}\psi(x, t, T) \tag{2.14}$$

for convenience.

2.1. Reduction independent of *T*. We reduce (2.11) by setting all the fields independent of T. This obviously means that

$$\epsilon = 0, \qquad \Omega = a_0, \tag{2.15}$$

and the system reduces to

$$\begin{gathered}
2MA_1\left(\beta_{xx} + a_0\beta - \frac{1}{M}\right)_x - A_2[M\beta_{xx} + a_0\beta + M]_x = 0, \\
M_t = 3M\beta_x - \beta M_x.
\end{gathered} \tag{2.16}$$

The reduction of the Lax pair can be obtained by setting

$$\psi_T = \lambda\psi.$$

In this case, (2.13) reduces to the third-order spectral problem

$$A_1\left(\psi_{xxx} + a_0\psi_x - \frac{\lambda}{M}\psi\right) + A_2\left(\psi_{xxx} + 2\frac{M_x}{M}\psi_{xx} + \frac{1}{M}(M_{xx} + a_0)\psi_x - \frac{\lambda}{M}\psi\right) = 0, \tag{2.17}$$

and its compatibility with (2.12) yields

$$\begin{gathered}
A_1[\lambda\psi_t + \psi_{xx} + \lambda\beta\psi_x + (a_0 - \lambda\beta_x)\psi] + \\
+ A_2\big[\lambda\psi_t + M\psi_{xx} + (\lambda\beta + M_x)\psi_x + (a_0 + \lambda\beta_x)\psi\big] = 0.
\end{gathered} \tag{2.18}$$

2.1.1. Degasperis–Procesi equation. For the case $A_1 = 1$ and $A_2 = 0$, we can integrate (2.16) as

$$\begin{aligned} &\beta_{xx} + a_0\beta = \frac{1}{M} + q_0, \\ &(\beta_{xx} + a_0\beta)_t + \beta\beta_{xxx} + 3\beta_x\beta_{xx} + 4a_0\beta\beta_x - 3q_0\beta_x = 0. \end{aligned} \tag{2.19}$$

For $q_0 = 0$ and $a_0 = -1$, (2.19) is the well-known Degasperis–Procesi equation [7], whose Lax pair in accordance with (2.17) and (2.18) is

$$\begin{aligned} &\psi_{xxx} - \psi_x - \lambda(\beta_{xx} - \beta)\psi = 0, \\ &\lambda\psi_t + \psi_{xx} + \lambda\beta\psi_x - (1 + \lambda\beta_x)\psi = 0, \end{aligned} \tag{2.20}$$

which is equivalent to the Lax pair in [7].

2.1.2. Vakhnenko equation. For the case $A_1 = 0$, $A_2 = 1$, and $a_0 = 0$, we can integrate (2.16) as

$$\begin{aligned} &\beta_{xx} + 1 = \frac{q_0}{M}, \\ &[(\beta_t + \beta\beta_x)_x + 3\beta]_x = 0, \end{aligned} \tag{2.21}$$

which is the derivative of the Vakhnenko equation [11], whose Lax pair in accordance with (2.17) and (2.18) is

$$\begin{aligned} &\psi_{xxx} + 2\frac{M_x}{M}\psi_{xx} + \frac{M_{xx}}{M}\psi_x - \frac{\lambda}{M}\psi = 0, \\ &\lambda\psi_t + M\psi_{xx} + (\lambda\beta + M_x)\psi_x + \lambda\beta_x\psi = 0. \end{aligned} \tag{2.22}$$

So far as we know, this is the first time a Lax pair for the Vakhnenko equation has been obtained in its original variables x and t. The previously known Lax pair [14], [12] was written after a reciprocal transformation that is a particular case of the one in the next section.

3. Second reciprocal transformation

A different reciprocal transformation can be constructed using the changes

$$\begin{aligned} &dx_2 = \eta(y,z,T)(dz - u(y,z,T)\,dy - \omega(y,z,T)\,dT), \\ &x_1 = y, \qquad x_3 = T. \end{aligned} \tag{3.1}$$

The partial derivatives then transform as

$$\frac{\partial}{\partial x_1} = \frac{\partial}{\partial y} + u\frac{\partial}{\partial z}, \qquad \frac{\partial}{\partial x_2} = \frac{1}{\eta}\frac{\partial}{\partial z}, \qquad \frac{\partial}{\partial x_3} = \frac{\partial}{\partial T} + \omega\frac{\partial}{\partial z}, \tag{3.2}$$

and the compatibility conditions

$$\begin{aligned} &\eta_y + (u\eta)_z = 0, \\ &\eta_T + (\eta\omega)_z = 0, \\ &u_T - \omega_y - u\omega_z + \omega u_z = 0 \end{aligned} \tag{3.3}$$

trivially arise. We select the transformation by setting the field H as the new independent variable z:

$$z = H(x_1, x_2, x_3) \quad \Rightarrow \quad dz = H_{x_1}\, dx_1 + H_{x_2}\, dx_2 + H_{x_3}\, dx_3. \tag{3.4}$$

Comparing (3.2) and (3.4), we obtain

$$\begin{aligned} H_{x_2}(x_1, x_2, x_3) &= \frac{1}{\eta(y = x_1, z = H, T = x_3)}, \\ H_{x_1}(x_1, x_2, x_3) &= u(y = x_1, z = H, T = x_3), \\ H_{x_3}(x_1, x_2, x_3) &= \omega(y = x_1, z = H, T = x_3). \end{aligned} \tag{3.5}$$

With this transformation, system (1.5) becomes

$$\begin{aligned} &A_1\left(u_{zy} + uu_{zz} + \frac{1}{4}u_z^2 + 3u - G\right)_z + A_2(u_{zy} + uu_{zz} + u_z^2 + 3u - G)_z = 0, \\ &G_y = (\omega - uG)_z, \end{aligned} \tag{3.6}$$

where $G(z, y, T)$ is defined as $G = \eta\Omega$ and we use (3.3) and (2.8). Lax pair (1.4) can also be transformed accordingly with the result

$$A_1\left[\Phi_{zy} + u\Phi_{zz} + \Phi + \frac{1}{2}u_z\Phi_z\right] + A_2[\Phi_{zy} + u\Phi_{zz} + \Phi] = 0$$

for the spatial part and

$$\begin{aligned} &A_1\left[\Phi_T - \Phi_{yyy} - u^3\Phi_{zzz} + 3u\left(u_y - \frac{1}{2}uu_z\right)\Phi_{zz} + \frac{3}{2}u_y\Phi\right] + \\ &\quad + A_1\left(\omega - u_{yy} + \frac{u_yu_z - u^2u_{zz} - uG - 3u^2}{2} - \frac{uu_z^2}{8}\right)\Phi_z + \\ &\quad + A_2(\Phi_T - \Phi_{yyy} - u^3\Phi_{zzz} + 3uu_y\Phi_{zz} + \Phi) + \\ &\qquad + A_2(\omega - 2uG - u_{yy} + u^2u_{zz} - u_yu_z + uu_z^2 + 3u^2)\Phi_z = 0 \end{aligned} \tag{3.7}$$

for the temporal part.

3.1. Reduction independent of T. As in Sec. 2, the reduction independent of T can be obtained by setting

$$\omega = 0, \qquad \Phi_T = \lambda\Phi. \tag{3.8}$$

System (3.6) reduces to

$$\begin{aligned} &G_y + (uG)_z = 0, \\ &A_1\left[u_{zy} + uu_{zz} + \frac{1}{4}u_z^2 + 3u - G\right]_z + A_2[u_{zy} + uu_{zz} + u_z^2 + 3u - G]_z = 0. \end{aligned} \tag{3.9}$$

It is very interesting that (3.9) for $G = 0$ reduces to the Vakhnenko equation up to a constant if $A_1 = 0$ and to a modified Vakhnenko equation if $A_2 = 0$.

The Lax pair can be reduced by directly applying (3.8) to (3.7), but it requires voluminous computations, which we have done with MAPLE. The result of these computations is

$$\begin{aligned} & A_1\left[\lambda\Phi_{zzz} + \left(G - \lambda\frac{N_z}{N}\right)\Phi_{zz} + \left(\frac{3}{2}G_z - \frac{N_z}{N}G\right)\Phi_z\right] + \\ & \quad + A_1\left[\left(G_{zz} + \frac{N^2}{2} - \frac{N_z}{N}G_z\right)\frac{\Phi}{2}\right] + \\ & \quad\quad + A_2[\lambda\Phi_{zzz} + G\Phi_{zz} + G_z\Phi_{zz} + (u_{zz}+1)\Phi] = 0, \\ & A_1\left[\Phi_y - \frac{2\lambda}{N}\Phi_{zz} + \left(u - \frac{2G}{N}\right)\Phi_z - \left(\frac{u_z}{2} + \frac{G_z}{N}\right)\Phi\right] + \\ & \quad\quad + A_2[\Phi_y - \lambda\Phi_{zz} + (u-G)\Phi_z - u_z\Phi] = 0, \end{aligned} \tag{3.10}$$

where we set $N = u_{zz} + 2$ for simplicity.

We note that for $G = 0$ and $A_1 = 0$, we obtain the 1+1 Lax pair

$$\begin{aligned} & \lambda\Phi_{zzz} + (u_{zz}+1)\Phi = 0, \\ & \Phi_y - \lambda\Phi_{zz} + u\Phi_z - u_z\Phi = 0, \end{aligned} \tag{3.11}$$

whose compatibility condition is

$$[u_{zy} + uu_{zz} + u_z^2 + 3u]_z = 0, \tag{3.12}$$

which is the derivative of Vakhnenko equation. Therefore, (3.11) is an alternative form of writing a Lax pair for the Vakhnenko equation.

4. Conclusions

We have presented a spectral problem in 2+1 dimensions and investigated different reciprocal transformations for it. We gave reductions to 1+1 dimensions of the transformed spectral problems. The Vakhnenko–Parkes and Degasperis–Procesi equations appear in this context as particular cases of (1.5).

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