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# 1 + 1 spectral problems arising from the Manakov–Santini system

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## Abstract
This paper deals with the spectral problem of the Manakov–Santini system. The point Lie symmetries of the Lax pair have been identified. Several similarity reductions arise from these symmetries. An important benefit of our procedure is that the study of the Lax pair instead of the partial differential equations yields the reductions of the eigenfunctions and also the spectral parameter. Therefore, we have obtained five interesting spectral problems in 1 + 1 dimensions.

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## 1. Introduction

One of the most powerful instruments to study and/or solve a given differential equation is the identification of the Lie point symmetries of the equation [8]. A standard method for finding solutions of a partial differential equation (PDE) is that of reduction by using Lie symmetries: each Lie symmetry allows a reduction of the PDE to a new equation with the number of independent variables reduced by one. Classical [8, 13] and nonclassical [2, 11] Lie symmetries are the usual way used to identify the reductions. The similarity reduction of PDEs obtained through the calculation of their Lie symmetries is a standard procedure that has been successfully applied in the scientific literature for many decades. The connection between these methods and/or other methods for obtaining similarity reductions has also been extensively discussed [5, 6].

As is well known, one of the best proofs of the integrability of a PDE is the existence of a Lax pair, which requires the introduction of a new dependent field (the eigenfunction) and a constant (the spectral parameter), which can also be interpreted as a new independent variable such that only the eigenfunction depends on it.

It is natural to deal with the problem of identifying reductions of the Lax pair instead of those of the equations [9]. The obvious benefit is that, in this case, we know how the

eigenfunction and spectral parameter will reduce [7]. This is by no means a trivial question, as we can see in the example that we are interested here. This example is the Manakov–Santini system [10], which reads

$$u_{xt} + u_{yy} + (uu_x)_x + v_x u_{xy} - v_y u_{xx} = 0$$
$$v_{xt} + v_{yy} + uv_{xx} + v_x v_{xy} - v_y v_{xx} = 0, \tag{1.1}$$

with the Lax pair

$$\psi_y = -(\lambda + v_x)\psi_x + u_x \psi_\lambda$$
$$\psi_t = -(\lambda^2 + \lambda v_x + u - v_y)\psi_x + (\lambda u_x - u_y)\psi_\lambda, \tag{1.2}$$

where $u = u(x, y, t)$, $v = v(x, y, t)$, $\psi = \psi(x, y, t, \lambda)$.

Equation (1.1) is a member of the Manakov–Santini hierarchy [10] and it is well known that it has several interesting reductions [3, 4, 12].

According to our previous statement, here we shall address with the Lie point symmetries of the Lax pair (1.2), where $u$, $v$, $\psi$ are the dependent variables and $x$, $y$, $t$, $\lambda$ the independent ones.

The plan of the paper is as follows.

- Calculation of the Lie symmetries of the Lax pair (1.2) is dealt with in section 2.
- The five different reduced spectral problems appear in section 3. The equations obtained from these spectral problems are also identified. Two of them are linear equations than can be integrated. The other three systems include equations such as the Monge–Ampere and modified Hunter–Saxton equations.
- We conclude with a section of conclusions.
- Some necessary but tedious expressions are listed in the appendix.

## 2. Lie point symmetries of the spectral problem

Here, we are interested in the Lie symmetries of the Lax pair. Actually, the symmetries of equations (1.1) are interesting in themselves, but we also wish to know how the eigenfunction and the spectral parameter transform under the action of a Lie symmetry. More precisely, we wish to know what these fields look like under the reduction associated with each symmetry. This is why we shall proceed to write the infinitesimal Lie point transformation of the variables and fields that appear in the spectral problem. The benefits of such a procedure have been shown in [9] and [7].

In the present case, it is important to note that the spectral parameter appears as an independent variable.

The infinitesimal form of the Lie point symmetry that we are considering is

$$x' = x + \varepsilon\,\xi_1(x, y, t, \lambda, \psi, u, v) + O(\varepsilon^2)$$
$$y' = y + \varepsilon\,\xi_2(x, y, t, \lambda, \psi, u, v) + O(\varepsilon^2)$$
$$t' = t + \varepsilon\,\xi_3(x, y, t, \lambda, \psi, u, v) + O(\varepsilon^2)$$
$$\lambda' = \lambda + \varepsilon\,\xi_4(x, y, t, \lambda, \psi, u, v) + O(\varepsilon^2)$$
$$u' = u + \varepsilon\,\phi_1(x, y, t, \lambda, \psi, u, v) + O(\varepsilon^2)$$
$$v' = v + \varepsilon\,\phi_2(x, y, t, \lambda, \psi, u, v) + O(\varepsilon^2)$$
$$\psi' = \psi + \varepsilon\,\phi_3(x, y, t, \lambda, \psi, u, v) + O(\varepsilon^2),$$

where $\epsilon$ is the group parameter. The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form

$$X = \xi_1 \frac{\partial}{\partial x} + \xi_2 \frac{\partial}{\partial y} + \xi_3 \frac{\partial}{\partial t} + \xi_4 \frac{\partial}{\partial \lambda} + \phi_1 \frac{\partial}{\partial u} + \phi_2 \frac{\partial}{\partial v} + \phi_3 \frac{\partial}{\partial \psi}. \tag{2.1}$$

We also need to know how the derivatives of the fields transform under the Lie symmetry, which means that we have to introduce the 'prolongations' of the action of the group to the different derivatives that appear in (1.2). Exactly how to calculate the prolongations is a very well-known procedure whose technical details can be found in [2, 13].

It is therefore necessary that the Lie transformation should leave (1.2) invariant. This yields an overdetermined system of equations for the infinitesimals $\xi_1(x, y, t, \lambda, \psi, u, v)$, $\xi_2(x, y, t, \lambda, \psi, u, v)$, $\xi_3(x, y, t, \lambda, \psi, u, v)$, $\xi_4(x, y, t, \lambda, \psi, u, v)$, $\phi_1(x, y, t, \lambda, \psi, u, v)$, $\phi_2(x, y, t, \lambda, \psi, u, v)$ and $\phi_3(x, y, t, \lambda, \psi, u, v)$.

Below is a summary of the classical Lie method [13] of finding Lie symmetries.

- Calculation of the prolongations of the derivatives of the fields that appear in (1.2).
- Substitution of the transformed fields (2.1) and their derivatives in (1.2).
- Set all the coefficients in $\epsilon$ at 0.
- Substitution of the prolongations.
- $\psi_y$ and $\psi_t$ can be substituted by using (1.2).
- The system of equations for the infinitesimals can be obtained by setting each coefficient in the different remaining derivatives of the fields at zero.

From the technical point of view, calculation of the determining equations can be performed by using computer packages such as MACSYMA or MAPLE. We have used both independently to determine the equations and solve them. The result is the following set of symmetries:

$$\xi_1 = -\frac{1}{2} (2\alpha - \tau_t)_t \, y^2 + \beta y + (2\alpha - \tau_t) \, x + \gamma$$

$$\xi_2 = \alpha y + \delta$$

$$\xi_3 = \tau$$

$$\xi_4 = (\alpha - \tau_t)_t \, y + (\alpha - \tau_t) \, \lambda - \beta - \delta_t$$

$$\phi_1 = 2 \, (\alpha - \tau_t) \, u + \frac{1}{2} \, (\alpha - \tau_t)_{tt} \, y^2 - (\beta + \delta_t)_t \, y - (\alpha - \tau_t)_t \, x + \theta \tag{2.2}$$

$$\phi_2 = (3\alpha - 2\tau_t) \, v + \left( \frac{\alpha}{2} - \frac{\tau_t}{3} \right)_{tt} y^3 - \left( \beta + \frac{\delta_t}{2} \right)_t y^2$$
$$+ [(2\tau_t - 3\alpha)_t x + \theta - \gamma_t] y + (2\beta + \delta_t) x + \sigma$$

$$\phi_3 = \Omega(\psi),$$

where $\Omega$ is an arbitrary function of $\psi$ and $\tau, \alpha, \beta, \delta, \gamma, \theta, \sigma$ are arbitrary functions of $t$. Therefore, the symmetries depend on eight arbitrary functions.

## 2.1. Nonclassical Lie symmetries

As is well known, there exist the so-called nonclassical symmetries [2, 11] that are symmetries of the equation together with the 'invariant surface conditions'

$$\phi_1 = \xi_1 u_x + \xi_2 u_y + \xi_3 u_t$$
$$\phi_2 = \xi_1 v_x + \xi_2 v_y + \xi_3 v_t \tag{2.3}$$
$$\phi_3 = \xi_1 \psi_x + \xi_2 \psi_y + \xi_3 \psi_t + \xi_4 \psi_\lambda.$$

These conditions allow us to eliminate more derivatives of the fields in the determining equations. As is well known, this elimination differs, depending on whether the values of $\xi_1, \xi_2, \xi_3$ are zero or not. We looked for these symmetries in (1.2) but all of them are contained in (2.2). Therefore, the nonclassical method does not provide new symmetries.

Let us now determine the $1 + 1$ spectral problems in $1 + 1$ dimensions derived from the different possible reductions of (2.2). These reductions can be obtained by solving the characteristic system [7, 13] associated with the vector field (2.1). In our case this system is

$$\frac{\mathrm{d}x}{\xi_1} = \frac{\mathrm{d}y}{\xi_2} = \frac{\mathrm{d}t}{\xi_3} = \frac{\mathrm{d}\lambda}{\xi_4} = \frac{\mathrm{d}u}{\phi_1} = \frac{\mathrm{d}v}{\phi_2} = \frac{\mathrm{d}\psi}{\phi_3}. \tag{2.4}$$

## 3. Similarity reductions of the spectral problem

There are several independent reductions depending on whether the arbitrary functions that appear in (2.2) are zero or not. We will classify the reductions into five classes.

*I. Reductions for $\tau \neq 0$.*

We can solve the characteristic equation (2.4) and we find the following results.

- *Reduced variables*. The reduced variables $z_1, z_2$ can be defined as

$$z_1 = B_1 y - B_2, \qquad z_2 = B_1^2 \tau x - C_1 B_1^2 y^2 - B_1 B_3 y - B_4. \tag{3.1}$$

- *Spectral parameter*. Let $\Lambda$ be the reduced spectral parameter. Thus, it is obtained as

$$\Lambda = \tau B_1 \lambda + C_2 B_1 y + (\delta B_1 + B_3). \tag{3.2}$$

- *Reduced fields*.

$$\int \frac{\mathrm{d}\psi}{\Omega(\psi)} = \mathrm{e}^{\int \frac{\mathrm{d}t}{\tau(t)}} \Phi(z_1, z_2, \Lambda)$$

$$u(x, y, t) = \frac{F(z_1, z_2) + B_5}{\tau^2 B_1^2} + \frac{C_2 B_1 x + N_1 y^2 + N_2 y}{\tau B_1} \tag{3.3}$$

$$v(x, y, t) = \frac{H(z_1, z_2) + B_6}{\tau^2 B_1^3} + \left(\frac{N_3 B_1 y + N_4}{\tau B_1}\right) x + \frac{N_5 B_1^2 y^3 + N_6 B_1 y^2 + N_7 y}{\tau^2 B_1^2},$$

where $F(z_1, z_2)$ and $H(z_1, z_2)$ are the reduced fields and $\Phi(z_1, z_2)$ is the reduced eigenfunction.

The derivatives of the functions $B_i = B_i(t)$, $i = 1, \ldots, 6$, are related to the seven arbitrary functions $\tau, \alpha, \delta, \beta, \gamma, \theta, \sigma$. The explicit relations are shown in appendix A. $C_1 = C_1(t)$, $C_2 = C_2(t)$ and $N_i = N_i(t)$, $i = 1, \ldots, 7$, are auxiliary functions that are also explicitly written in appendix A.

- *Reduced spectral problem*. We can now substitute the reductions (3.1)–(3.3) in (1.2) to obtain the following Lax pair in $1 + 1$ dimensions:

$$\frac{\partial \Phi}{\partial z_1} + \frac{\partial \Phi}{\partial z_2}(H_{z_2} + \Lambda) - F_{z_2}\frac{\partial \Phi}{\partial \Lambda} = 0$$

$$(F_{z_1} - \Lambda F_{z_2})\frac{\partial \Phi}{\partial \Lambda} + (F - H_{z_1} + \Lambda H_{z_2} + \Lambda^2)\frac{\partial \Phi}{\partial z_2} + \Phi = 0. \tag{3.4}$$

- *Reduced equations*. The compatibility condition of (3.4) yields the following system of equations in $1 + 1$ dimensions:

$$(FF_{z_2} - H_{z_1}F_{z_2})_{z_2} + (F_{z_1} + H_{z_2}F_{z_2})_{z_1} = 0$$

$$H_{z_1 z_1} + FH_{z_2 z_2} + H_{z_2}H_{z_1 z_2} - H_{z_1}H_{z_2 z_2} = 0, \tag{3.5}$$

which contains as a particular case the equations

  – $H = 0$

$$F_{z_1 z_1} + FF_{z_2 z_2} = 0$$

  – $F = 0$

$$H_{z_1 z_1} + H_{z_2}H_{z_1 z_2} - H_{z_1}H_{z_2 z_2} = 0$$

  – $F = H_{z_1}$

$$\left(H_{z_1} + \frac{H_{z_2}^2}{2}\right)_{z_1 z_1} = 0.$$

*II. Reductions for $\tau = 0$, $\alpha \neq 0$.*

In this case it is useful to define the function

$$K = K(y, t) = y + \frac{\delta}{\alpha}.$$

- *Reduced variables*. Let $z_1$, $z_2$ be the reduced variables. They can be obtained by solving (2.4) as

$$z_1 = \int M_1(t)\, dt$$

$$z_2 = \frac{x - B_2}{K^2 M_1} - \frac{B_1}{K M_1} + \ln(K) - M_2. \tag{3.6}$$

- *Spectral parameter*. The reduced spectral parameter $\Lambda$ is

$$\Lambda = \frac{\lambda - B_3}{K M_1} - \ln(K) + z_2 - M_3. \tag{3.7}$$

- *Reduced fields*. The integration of (2.4) yields the following reductions for the fields:

$$\int \frac{d\psi}{\Omega(\psi)} = K^{\left(\frac{1}{\alpha}\right)} \Phi(z_1, z_2, \Lambda)$$

$$u(x, y, t) = \frac{dB_3}{dt} K + B_4 + K^2 M_1^2 \left[ F(z_1, z_2) + \frac{1}{2}(z_2 - \ln(K) + 1 - B_0)^2 + M_4 \right]$$

$$v(x, y, t) = -2[z_2 + M_2 - \ln(K)]K^2 M_1 N_1 + \frac{dN_1}{dt} K^2 + N_2 K + B_5 \tag{3.8}$$

$$+ K^3 M_1^2 \left[ H(z_1, z_2) + \frac{3}{2}\left(z_2 - \ln(K) + \frac{4}{3} - \frac{2}{3}B_0\right)^2 + M_5 \right],$$

where $F(z_1, z_2)$ and $H(z_1, z_2)$ are the reduced fields and $\Phi(z_1, z_2)$ is the reduced eigenfunction.

    Functions $B_i = B_i(t)$, $i = 0, \dots, 5$, and $N_i = N_i(t)$, $i = 1, \dots, 2$, are defined in terms of the six arbitrary functions $\alpha, \beta, \delta, \gamma, \theta, \sigma$. Their explicit expressions appear in appendix B. The five $M_i = M_i(t)$, $i = 1, \dots, 5$, functions are, in principle, arbitrary but we have fixed them in the forms that appear in appendix B in order to have the simplest form for the spectral problem.

- *Reduced spectral problem.* In this case, the Lax pair reduces to the following non-autonomous form:

$$\frac{\partial \Phi}{\partial z_1} + (F - 3H)\frac{\partial \Phi}{\partial z_2} + (\Lambda^2 - \Lambda + 3F - 3H)\frac{\partial \Phi}{\partial \Lambda} - e^{-z_1}(\Lambda + z_2)\Phi = 0$$

$$(H_{z_2} - F_{z_2})\frac{\partial \Phi}{\partial \Lambda} + (H_{z_2} + \Lambda)\frac{\partial \Phi}{\partial z_2} + e^{-z_1}\Phi = 0. \tag{3.9}$$

- *Reduced equations.* Although (3.9) is non-autonomous, it yields the following autonomous system:

$$H_{z_1 z_2} + (F - 3H)H_{z_2 z_2} + H_{z_2}^2 + H_{z_2} + 3(F - H) = 0$$

$$F_{z_1 z_2} + (F - 3H)F_{z_2 z_2} + F_{z_2}^2 - H_{z_2} + 2F_{z_2} + 3(F - H) = 0. \tag{3.10}$$

- When $F = H$, the system includes the equation

$$(H_{z_1} - 2H H_{z_2} + H)_{z_2} + 3H_{z_2}^2 = 0,$$

which can be understood as a modified Hunter–Saxton equation [1]. In this particular case, the Lax pair (3.9) can be written in the autonomous form:

$$(H_{z_2} + \Lambda)\frac{\partial \Psi}{\partial z_2} + H_{z_2 z_2}\Psi = 0$$

$$\frac{\partial \Psi}{\partial z_1} - 2H\frac{\partial \Psi}{\partial z_2} + \Lambda(\Lambda - 1)\frac{\partial \Psi}{\partial \Lambda} - (H_{z_2} - \Lambda)\Psi = 0$$

by means of the transformation $\Psi = (\ln(\Phi))_{z_2}$.

## III. Reductions for $\tau = \alpha = 0$, $\delta \neq 0$.

- *Reduced variables.* The integration of the characteristic system provides the reduced variables

$$z_1 = \int M_1 \, dt$$

$$z_2 = \frac{1}{M_2}\left(x - \frac{\beta y^2 + 2\gamma y}{2\delta}\right). \tag{3.11}$$

- *Spectral parameter.* The reduction of the spectral parameter is

$$\Lambda = \frac{\lambda \delta + B_1 y - M_3}{M_2}. \tag{3.12}$$

- *Reduced fields.* The reduced fields are

$$\int \frac{d\psi}{\Omega(\psi)} = e^{\left(\frac{y}{\delta} + M_4\right)}\Phi(z_1, z_2, \Lambda)$$

$$u(x, y, t) = \frac{M_2^2}{\delta^2}F(z_1 z_2) + \frac{-\frac{dB_1}{dt}y^2 + 2\theta y}{2\delta} + N_1 \tag{3.13}$$

$$v(x, y, t) = \frac{M_2^2}{\delta}H(z_1 z_2) + \frac{B_4 y^3 + B_3 y^2 + (2M_2 B_2 z_2 + \sigma)\, y}{\delta} + N_2,$$

where $F(z_1, z_2)$ and $H(z_1, z_2)$ are the reduced fields, $\Phi(z_1, z_2)$ is the reduced eigenfunction and $\Lambda$ is the reduced spectral parameter.

Functions $B_i = B_i(t), i = 0, \ldots, 4$, are defined in terms of the arbitrary functions $\beta, \delta, \gamma, \theta, \sigma$. Their explicit expressions appear in appendix C. The $M_i = M_i(t), i = 1, \ldots, 3$, $N_i = N_i(t, z_2), i = 1, \ldots, 3$, functions are in principle arbitrary, but we can fix them in the forms that appear in appendix C in order to have the simplest form for the spectral problem.

- *Reduced spectral problem*

$$(H_{z_2} + \Lambda)\frac{\partial \Phi}{\partial z_2} - F_{z_2}\frac{\partial \Phi}{\partial \Lambda} + \Phi = 0$$

$$\frac{\partial \Phi}{\partial z_1} + F\frac{\partial \Phi}{\partial z_2} - \Lambda \Phi = 0. \tag{3.14}$$

- *Reduced equations.* The compatibility condition yields

$$H_{z_1 z_2} + F H_{z_2 z_2} = 0$$

$$F_{z_1 z_2} + F F_{z_2 z_2} + F_{z_2}^2 = 0. \tag{3.15}$$

- It is interesting to note that the equation for $F$ is the non-dispersive KdV equation

$$(F_{z_1} + F F_{z_2})_{z_2} = 0.$$

By eliminating $F$ between the two equations (3.15) for $H$ we obtain the equation

$$F = -\frac{H_{z_1 z_2}}{H_{z_2 z_2}}$$

$$\left[\frac{1}{H_{z_2 z_2}}\left(\frac{H_{z_1 z_2}^2 - H_{z_1 z_1}H_{z_2 z_2}}{H_{z_2 z_2}}\right)_{z_2}\right]_{z_2} = 0,$$

which can be integrated twice with respect to $z_2$. It yields the generalized Monge–Ampere equation

$$H_{z_1 z_2}^2 - H_{z_1 z_1}H_{z_2 z_2} = a(z_1)H_{z_2} + b(z_1).$$

*IV. Reductions for $\tau = \alpha = \delta = 0$, $\beta \neq 0$.*

- *Reduced variables.* The integration of the characteristic system allows us to write the reduced variables as

$$z_1 = \int \beta(t)\, dt$$

$$z_2 = y + \frac{\gamma(t)}{\beta(t)}. \tag{3.16}$$

- *Spectral parameter.* The reduced spectral parameter $\Lambda$ is

$$\Lambda = \frac{\lambda z_2 + x}{\beta z_2}. \tag{3.17}$$

- *Reduced fields.* The reduced fields are

$$\int \frac{d\psi}{\Omega(\psi)} = e^{\frac{\lambda}{\beta}}\, \Phi(z_1, z_2, \Lambda)$$

$$u(x, y, t) = (B_2 - B_1\, y)\frac{\beta\, x}{z_2} + \beta^2\, F(z_1, z_2) \tag{3.18}$$

$$v(x, y, t) = (-B_1\, y^2 + (B_2 - B_3)y + B_4)\frac{\beta\, x}{z_2} + \frac{x^2}{z_2} + \beta^2\, H(z_1, z_2),$$

where $F(z_1, z_2)$ and $H(z_1, z_2)$ are the reduced fields and $\Phi(z_1, z_2)$ is the reduced eigenfunction.

The functions $B_i = B_i(z_1), i = 0, \ldots, 4$, are defined in terms of the four arbitrary functions $\beta, \gamma, \theta, \sigma$. Their explicit expressions appear in appendix D.

- *Reduced spectral problem.*

$$\frac{\partial \Phi}{\partial z_1} + \left(\frac{dB_0}{dz_1} - \Lambda\right)\frac{\partial \Phi}{\partial z_2} + \left[F_{z_2} - \Lambda B_1 + \frac{1}{z_2}\left(F - H_{z_2}\right)\right]\frac{\partial \Phi}{\partial \Lambda} + (F_{z_2} - \Lambda B_1)\Phi = 0$$
$$\frac{\partial \Phi}{\partial z_2} + \frac{1}{z_2}\left(\Lambda - \frac{dB_0}{dz_1} + \frac{B_5}{z_2}\right)\frac{\partial \Phi}{\partial \Lambda} + \left(B_1 - \frac{B_6}{z_2}\right)\Phi = 0. \tag{3.19}$$

- *Reduced equations.* The compatibility condition yields the linear equations

$$\frac{d^2 H}{dz_2^2} + \frac{2}{z_2}\left(F - \frac{dH}{dz_2}\right) = \frac{B_5^2}{z_2^3} + \frac{B_6 B_5}{z_2^2} - \frac{B_1 B_5 + \frac{dB_5}{dz_1}}{z_2} + \frac{d^2 B_0}{dz_1^2} + B_1\frac{dB_0}{dz_1} - \frac{dB_6}{dz_1} + z_2\frac{dB_1}{dz_1},$$
$$\frac{d^2 F}{dz_2^2} = \frac{B_5 B_6}{z_2^3} - \frac{1}{z_2}\frac{dB_6}{dz_1} + \frac{dB_1}{dz_1},$$

which can easily be integrated as

$$H(z_1, z_2) = \frac{dB_6}{dz_1}\left(1 - \ln(z_2)\right)z_2^2 + \frac{A_2(z_1)z_2^3}{2} + \frac{1}{2}\left(2A_1(z_1) - B_1\frac{dB_0}{dz_1} - \frac{d^2 B_0}{dz_1^2}\right)z_2^2$$
$$+ \left(2C_1 + \frac{dB_5}{dz_1} + B_1 B_5\right)\frac{z_2}{2} + \frac{B_5^2}{4z_2} + C_2, \tag{3.20}$$

$$F(z_1, z_2) = \frac{dB_6}{dz_1}z_2\left(1 - \ln(z_2)\right) + \frac{B_5 B_6}{2z_2} + \frac{z_2^2}{2}\frac{dB_1}{dz_1} + A_1 z_2 + C_1,$$

where $A_1, A_2, C_1, C_2$ are arbitrary functions of $z_1$.

*V. Reductions for $\tau = \alpha = \delta = \beta = 0, \gamma \neq 0$.*

- *Reduced variables.* The reduced variables $z_1$ and $z_2$ are

$$z_1 = \int \gamma(t)\, dt$$
$$z_2 = y. \tag{3.21}$$

- *Spectral parameter.* The reduced spectral parameter $\Lambda$ is

$$\Lambda = \frac{\lambda}{\gamma}. \tag{3.22}$$

- *Reduced fields.* The reduction of the fields is

$$\int \frac{d\psi}{\Omega(\psi)} = e^{\frac{x}{\gamma}}\,\Phi(z_1, z_2, \Lambda)$$
$$u(x, y, t) = \gamma B_1 x + \gamma^2 F(z_1 z_2) \tag{3.23}$$
$$v(x, y, t) = \gamma(B_2 y + B_3)x + \gamma^2 H(z_1, z_2),$$

where $F(z_1, z_2)$ and $H(z_1, z_2)$ are the reduced fields and $\Phi(z_1, z_2)$ is the reduced eigenfunction.

The functions $B_i = B_i(z_1)$, $i = 0, \ldots, 3$, are defined in terms of the arbitrary functions $\gamma, \theta, \sigma$. Their explicit expressions appear in appendix E.

- *Reduced spectral problem.*

$$\frac{\partial \Phi}{\partial z_1} - \Lambda\frac{\partial \Phi}{\partial z_2} + [F_{z_2} + \Lambda(B_2 - B_1)]\frac{\partial \Phi}{\partial \Lambda} + \left(F - H_{z_2}\right)\Phi = 0$$
$$\frac{\partial \Phi}{\partial z_2} - B_1\frac{\partial \Phi}{\partial \Lambda} + (B_2 z_2 + B_3 + \Lambda)\Phi = 0. \tag{3.24}$$

- *Reduced equations*. The compatibility condition yields the linear equations

$$\frac{d^2 F}{dz_2^2} = B_1 B_2 - 2B_1^2 - \frac{dB_1}{dz_1},$$

$$\frac{d^2 H}{dz_2^2} = -\left(B_1 B_2 + \frac{dB_2}{dz_1}\right) z_2 - \left(B_1 B_3 + \frac{dB_3}{dz_1}\right),$$

which can easily be integrated as

$$F(z_1, z_2) = \left(B_1 B_2 - 2B_1^2 - \frac{dB_1}{dz_1}\right) \frac{z_2^2}{2} + A_1 z_2 + C_1,$$

$$H(z_1, z_2) = \left(B_1 B_2 + \frac{dB_2}{dz_1}\right) \frac{z_2^3}{6} - \left(B_1 B_3 + \frac{dB_3}{dz_1}\right) \frac{z_2^2}{2} + A_2 z_2 + C_2,$$

(3.25)

where $A_1$, $A_2$, $C_1$, $C_2$ are arbitrary functions of $z_1$.

## 4. Conclusions

- We have studied the Lie symmetries of the spectral problem of the Manakov–Santini equation. The procedure requires the consideration of the spectral parameter as an additional independent variable. Therefore, the Lax pair would be considered as a system with three fields and four independent variables.
- We have used computer packages such as MAPLE and MACSYMA to handle the calculation. The resulting symmetries depend on seven arbitrary functions of $t$ and one arbitrary function of $\psi$.
- We also looked for nonclassical symmetries and have realized that they are no different from the classical ones.
- Five independent reductions arise from the symmetries identified. The spectral problems are obtained for all the reductions. Two of them give rise to nonlinear equations that can be easily integrated. The other three yield reduced systems that include non-dispersive KdV, generalized Monge–Ampere and modified Hunter–Saxton equations, among others.

## Appendix A.

The functions $B_i(t)$, $C_i(t)$, $N_i(t)$ that appear in equations (3.1)–(3.3) are

$$\frac{dB_1}{dt} = -\frac{\alpha B_1}{\tau}$$

$$\frac{dB_2}{dt} = \frac{\delta B_1}{\tau}$$

$$\frac{dB_3}{dt} = \beta B_1 - 2C_1 \frac{dB_2}{dt}$$

$$\frac{dB_4}{dt} = \gamma B_1^2 - B_3 \frac{dB_2}{dt}$$

$$\frac{dB_5}{dt} = B_1^2 \tau \theta - \delta B_1 N_2 - B_1^2 c_2 \gamma$$

$$\frac{\mathrm{d}B_6}{\mathrm{d}t} = \sigma\tau B_1^3 - N_7\frac{\mathrm{d}B_2}{\mathrm{d}t} - \gamma N_4 B_1^2$$

$$C_1(t) = \frac{\tau_t}{2} - \alpha$$

$$C_2(t) = \tau_t - \alpha$$

$$N_1 = -\frac{1}{2}\frac{\mathrm{d}(C_2 B_1)}{\mathrm{d}t}$$

$$N_2 = -\frac{\mathrm{d}(B_3 + \delta B_1)}{\mathrm{d}t}$$

$$N_3 = C_2 + 2C_1$$

$$N_4 = 2B_3 + \delta B_1$$

$$N_5 = -\frac{1}{6}\frac{\mathrm{d}(\tau N_3)}{\mathrm{d}t} + \frac{1}{2}\alpha N_3$$

$$N_6 = -\frac{1}{2}\frac{\mathrm{d}(\tau N_4)}{\mathrm{d}t} + \alpha N_4$$

$$N_7 = B_5 - B_3^2 - \tau\gamma B_1^2.$$

## Appendix B.

In equations (3.6)–(3.8), several functions $B_i(t)$, $M_i(t)$, $N_i(t)$ of $t$ appear. They are defined as

$$B_0 = \frac{1}{M_1^2}\frac{\mathrm{d}M_1}{\mathrm{d}t}$$

$$B_1 = \frac{-\beta - 2\delta M_1}{\alpha}$$

$$B_2 = \frac{\delta^2 M_1 + \delta\beta - \gamma\alpha}{2\alpha^2}$$

$$B_3 = \frac{1}{\alpha}\left(\delta M_1 + \beta + \frac{\mathrm{d}\delta}{\mathrm{d}t}\right)$$

$$B_4 = \frac{1}{4\alpha^2}\left(\beta\delta M_1 - \alpha\gamma M_1 - \delta^2\frac{\mathrm{d}M_1}{\mathrm{d}t} - 2\frac{\mathrm{d}^2\delta}{\mathrm{d}t^2} - 2\delta\frac{\mathrm{d}\beta}{\mathrm{d}t}\right) - \frac{\theta}{2\alpha}$$

$$B_5 = \frac{\delta}{6\alpha^3}\left(\delta^2\frac{\mathrm{d}M_1}{\mathrm{d}t} - 2\delta^2 M_1^2 + 2\delta\frac{\mathrm{d}\beta}{\mathrm{d}t} + \delta\frac{\mathrm{d}^2\delta}{\mathrm{d}t^2} - 5\beta\delta M_1 - 2\beta^2 - \delta M_1\frac{\mathrm{d}\delta}{\mathrm{d}t} - \beta\frac{\mathrm{d}\delta}{\mathrm{d}t}\right)$$
$$\qquad + \frac{1}{6\alpha^2}\left(2\beta\gamma + 2\delta\theta + 3\delta\gamma M_1 + \gamma\frac{\mathrm{d}\delta}{\mathrm{d}t} - 2\delta\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right) - \frac{\sigma}{3\alpha}$$

$$N_1 = -\frac{1}{2}(B_3 - B_1)$$

$$N_2 = B_4 - B_1 B_3 - \frac{\mathrm{d}B_2}{\mathrm{d}t}$$

$$M_1 = \frac{1}{\alpha}\frac{\mathrm{d}\alpha}{\mathrm{d}t}$$

$$M_2 = \frac{3 - B_0}{2}$$

$$M_3 = B_0 - 2$$

$$M_4 = \frac{1}{2}\left(\frac{1}{M_1}\frac{\mathrm{d}B_0}{\mathrm{d}t} - B_0\right)$$

$$M_5 = \frac{1}{3}\left(\frac{1}{M_1}\frac{\mathrm{d}B_0}{\mathrm{d}t} - B_0\right) + \frac{1}{6}.$$

## Appendix C.

The functions $B_i(t)$, $M_i(t)$ and $N_i(t, z_2)$ that appear in (3.11)–(3.13) are

$$
\begin{aligned}
B_1 &= \beta + \frac{\mathrm{d}\delta}{\mathrm{d}t} \\
B_2 &= \beta + \frac{1}{2}\frac{\mathrm{d}\delta}{\mathrm{d}t} \\
B_3 &= \frac{1}{2}\left(\theta - \frac{\mathrm{d}\gamma}{\mathrm{d}t} + 2\frac{\gamma}{\delta}B_2\right) \\
B_4 &= \frac{1}{3}\left(\frac{\beta B_2}{\delta} - \frac{\mathrm{d}B_2}{\mathrm{d}t}\right) \\
M_1 &= \frac{M_2}{\delta^2} \\
M_2 &= \mathrm{e}^{\left(-\int\frac{\beta}{\delta}\mathrm{d}t\right)} \\
\frac{\mathrm{d}M_3}{\mathrm{d}t} &= \theta - \frac{\beta M_3 + \beta\gamma + \gamma\frac{\mathrm{d}\delta}{\mathrm{d}t}}{\delta} \\
\frac{\mathrm{d}M_4}{\mathrm{d}t} &= \frac{\gamma + M_3}{\delta^2} \\
N_1 &= \frac{M_2}{\delta}B_1 z_2 + \frac{\sigma}{\delta} - \frac{\gamma^2}{\delta^2} \\
N_2 &= \frac{M_2(\gamma - M_3)}{\delta}z_2.
\end{aligned}
\tag{C.1}
$$

## Appendix D.

In (3.16)–(3.18) several functions $B_i(z_1)$ are introduced. These read

$$
\begin{aligned}
B_0 &= \left(\frac{\gamma}{\beta^2}\right)_{[t=t(z_1)]} \\
B_1 &= \left(\frac{\beta_t}{\beta^2}\right)_{[t=t(z_1)]} \\
B_2 &= \left(\frac{\theta}{\beta^2}\right)_{[t=t(z_1)]} \\
B_3 &= \left(\frac{\gamma_t}{\beta^2}\right)_{[t=t(z_1)]} \\
B_4 &= \left(\frac{\sigma}{\beta^2}\right)_{[t=t(z_1)]} \\
B_5 &= \left(B_4 - B_0 B_2 + \frac{1}{\beta}\frac{\mathrm{d}B_0}{\mathrm{d}t}\right)_{[t=t(z_1)]} \\
B_6 &= (B_2 + B_1 B_0)_{[t=t(z_1)]},
\end{aligned}
$$

where we have used $\frac{\mathrm{d}B_i}{\mathrm{d}t} = \beta\frac{\mathrm{d}B_i}{\mathrm{d}z_1}$ according to the definition of $z_1$ in (3.16).

## Appendix E.

The functions $B_i(z_1)$ introduced in (3.20)–(3.22) are

$$B_1 = \left( \frac{\theta + \gamma_t}{\gamma^2} \right)_{[t=t(z_1)]}$$

$$B_2 = \left( \frac{\theta}{\gamma^2} \right)_{[t=t(z_1)]}$$

$$B_3 = \left( \frac{\sigma}{\gamma^2} \right)_{[t=t(z_1)]}.$$

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