Continuum of extended states in the spectrum of a one-dimensional random potential

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We describe a one-dimensional disordered system, based on the Pöschl-Teller potential, that exhibits a continuum of extended states which is independent of the random or correlated character of the sequence and of the length of the system. The delocalization of the electronic states occurs in the whole positive spectrum where the system shows a perfect transmission.

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Since the work of Anderson[1] localization has been considered as a central tenet of the theory of disordered systems. Its significance as an unavoidable consequence of the presence of disorder in the systems was later enhanced by the results of the scaling theory[2] that originially predicted the localization of all electronic states for any degree of disorder in two-dimensional (2D) and one-dimensional (1D) structures while the existence of a metal to insulator transition (MIT) was permitted in three-dimensional (3D) systems.[3] It was subsequently shown that in several configurations of 2D electron and hole systems a MIT could occur as a function of the density of carriers[4] or as a result of an applied magnetic field.[5] Such experimental observations meant the reopening of the localization problem and the scaling theory was enlarged to describe these new results. In one-dimensional systems the appearance of disordered models exhibiting short-range and long-range correlations also showed that extended states can exist in the spectrum of a 1D disordered structure. Short-range correlations lead to the emergence of isolated extended states which constitute a zero measure set of the spectrum in the thermodynamic limit,[6–8] while long-range correlations give rise to the appearance of a phase of apparently extended states and therefore a qualitative MIT in 1D.[9] Correlations can alter the random character of the structures and improve noticeably their transport properties as it has been experimentally verified in different systems such as semiconductor superlattices[10] or microwave guides.[11] In this work we go one step further and describe a one-dimensional disordered system, based on the Pöschl-Teller potential, that exhibits a continuum of extended states which is independent of the random or correlated character of the sequence and of the length of the system. We then enlarge the family of disordered models showing a perfect transmission within a continuum energy interval, but this time the total transparency is provided by the potential itself and it is not due to the existence of statistical correlations in the disordered sequence.

Let us consider the general Pöschl-Teller potential, shown in Fig. 1, and given by

$$V(x) = \frac{\hbar^2\alpha^2}{2m}\frac{V}{\cosh^2(\alpha x)}. \tag{1}$$

It resembles the form of an atomic well or barrier depending on the sign of V, a dimensionless parameter that together with α determines the height or depth of the potential. The parameter α, with units of inverse of length, controls the half-width of the potential which reads $d_{1/2} = 2\alpha^{-1}\,\mathrm{arccosh}\,\sqrt{2}$. The larger α is, the narrower and deeper the potential becomes. The Schrödinger equation for the Pöschl-Teller potential is analytically solvable and its solutions are well known.[12,13] The asymptotic transmission matrix for this potential has been obtained previously by the authors[14] and it reads

$$\mathcal{M} = \begin{pmatrix} e^{i\varphi}\sqrt{1+w^2} & -iw \\ iw & e^{-i\varphi}\sqrt{1+w^2} \end{pmatrix}, \tag{2}$$

where

$$w = \frac{\sin(\pi b)}{\sinh(\pi k/\alpha)}, \quad b = \frac{1}{2} + \sqrt{\frac{1}{4} - V}, \tag{3}$$

$$\varphi = \frac{\pi}{2} + \arg\left\{\frac{\Gamma^2(ik/\alpha)}{\Gamma(b+ik/\alpha)\Gamma(1-b+ik/\alpha)}\right\}, \tag{4}$$

$k = \sqrt{2mE}/\hbar$ and $\Gamma(z)$ is the complex Euler gamma function, also w is always a real quantity as can be seen in its alternative definition $w = \cosh(\pi\sqrt{V-1/4})/\sinh(k\pi/\alpha)$. The dimensionless amplitude in terms of b reads $V = -b(b-1)$ which is the usual form found in the literature. Let us remark that the above expressions are only valid for positive energies [i.e., $k \in \mathbb{R}$]. From (2) the asymptotic probability of transmission is $T = (1+w^2)^{-1}$. To build a chain with the potentials described, one must do the approximation of considering that each potential unit has a finite range. Hence a cutoff must be included in the Pöschl-Teller potential. Using this approximation one obtains matrices suitable to be arranged in linear chains, applying the composition technique described in Ref. 14. Let us suppose that the potential is appreciable only in-

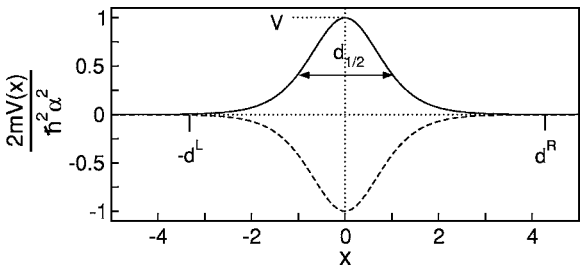

FIG. 1. Pöschl-Teller potential defined in Eq. (1).

side the interval $[-d^L, d^R]$, as shown in Fig. 1. Outside this interval the wave function is assumed to be a superposition of the free particle solutions. Then the transmission matrix for the cutoff potential reads

$$\mathbf{M} = \begin{pmatrix} e^{i[\varphi + k(d^R + d^L)]}\sqrt{1+w^2} & -iwe^{ik(d^R - d^L)} \\ iwe^{-ik(d^R - d^L)} & e^{-i[\varphi + k(d^R + d^L)]}\sqrt{1+w^2} \end{pmatrix}. \quad (5)$$

The cutoff matrix is the same as the asymptotic one plus an extra phase term in the diagonal elements that accounts for the total distance $(d^R + d^L)$ during which the particle feels the effect of the potential, and also an extra phase term in the off-diagonal elements measuring the asymmetry of the cutoff $(d^R - d^L)$. These phases are the key quantities since they will be responsible for the interference processes that produce the transmission patterns. In our case, due to the rapid decay of the Pöschl-Teller potential, the cutoff distance admits very reasonable values. In fact we have seen that for a sensible wide range of the parameters α and V, one can take as a minimum value for the cutoff distance $d_0 = 2d_{1/2} \simeq 3.5/\alpha$ where $d_{1/2}$ is the half-width. Taking $d^{L,R} \geqslant d_0$ the connection procedure works really well, as we have checked in all cases considered by comparing the analytical composition technique versus a numerical integration of the Schrödinger equation for the global potential. The above matrices can be used to obtain analytical expressions for the scattering amplitudes of different potential profiles including a few atoms resembling molecular structures.[14] In this work our main interest is to consider the transmission matrix (5) to make a continuous disordered model in the form of a large chain of these potentials with random parameters. Let us consider now the effects of uncorrelated disorder upon this particular model. From (5) one is led to the following canonical relation among the values of the electronic states at contiguous sites of the chain,

$$\Psi_{j+1} = \left(\bar{S}_j + S_{j-1}\frac{K_j}{K_{j-1}}\right)\Psi_j - \frac{K_j}{K_{j-1}}\Psi_{j-1}, \quad (6)$$

where

$$\bar{S}_j = -w_j \sin[k(d_j^L - d_j^R)] + \sqrt{1+w_j^2}\cos(\Phi_j), \quad (7)$$

$$S_j = w_j \sin[k(d_j^L - d_j^R)] + \sqrt{1+w_j^2}\cos(\Phi_j), \quad (8)$$

$$K_j = w_j \cos[k(d_j^L - d_j^R)] + \sqrt{1+w_j^2}\sin(\Phi_j), \quad (9)$$

in terms of w and φ defined in (3) and (4) and $\Phi_j = k(d_j^L + d_j^R) + \varphi_j$. The amplitudes Ψ_j correspond to the value of the state at the junction points of the potentials as shown in Fig. 2, and in this case each potential is determined by four parameters: d_j^L, d_j^R, α_j, and V_j. The form of the canonical relation obtained from the transmission matrix coincides with the Poincaré map derived by Sánchez and co-workers for one-dimensional potentials;[7] in fact, expression (6) is formally independent of the potential model. The canonical equation is essential to obtain the properties of the disordered system in the thermodynamic limit. From the canonical relation, the relevant quantities of the disordered composite Pöschl-Teller model such as density of states (DOS) and localization length can be numerically obtained in the thermodynamic limit by using the functional equation formalism, which has already been successfully applied to other disordered models by the authors.[8] The disordered compositions of Pöschl-Teller potentials give rise to the emergence of exciting properties such as fractal DOS, existence of different types of isolated extended states in the spectrum, and the appearance of bound states for the negative spectrum which can be completely delocalized. A thorough study of all these features will be reported elsewhere.[15] This work is devoted to describe the properties of the disordered system composed of a particular type of Pöschl-Teller potentials: the resonant wells. One characteristic feature of the potential (1) is that $T=1$ for all energies whenever b is a real integer. Hence an absolute resonant transmission occurs for potential wells with $V=-2,-6,-12,-20,\ldots$ independently of the value of α. The resonant wells correspond to potentials with an integer value of $b>1$. Since in this case $w=0$, the transmission matrix for a resonant well becomes diagonal and its nonzero elements are simply the phases $e^{\pm i\Phi_j}$, that is, it is the transmission matrix of a zero potential. The resonant well for positive energies behaves as a zero potential with an effective length $L_{\text{eff}}(k) \equiv \Phi_j/k = \varphi/k + (d^R + d^L)$ that depends on the energy. For a resonant well described by parameters $\{d_\gamma^L, d_\gamma^R, \alpha_\gamma, b_\gamma\}$ it can be proved by induction using the properties of the gamma function that the following expression holds,

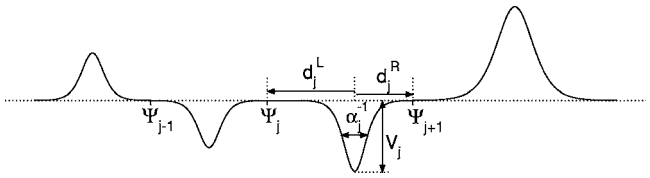

FIG. 2. Potential of a disordered Pöschl-Teller wire.

$$\epsilon L_{\text{eff}_\gamma}(\epsilon) = \epsilon\frac{\alpha_\gamma(d_\gamma^R + d_\gamma^L)}{(\alpha_\gamma/\alpha)} - 2\sum_{j=1}^{b_\gamma - 1}\arctan\left(\frac{\epsilon}{j(\alpha_\gamma/\alpha)}\right) + (b_\gamma - 1)\pi, \quad (10)$$

where $kL_{\text{eff}_\gamma}(k) \equiv \epsilon L_{\text{eff}_\gamma}(\epsilon)$, and the variable $\epsilon \equiv k/\alpha$ is a dimensionless representation of the energy, and α is the reference value for the parameters $\{\alpha_\gamma\}$. Now let us consider a disordered chain entirely composed of resonant wells with different parameters. For positive energies the functions appearing in the canonical equation of the system (6) reduce to $\bar{S}_j \equiv S_j = \cos[kL_{\text{eff}_j}(k)]$ and $K_j = \sin[kL_{\text{eff}_j}(k)]$. It can be easily checked that these functions define the canonical equation for a zero potential where the wave function is evaluated at different distances corresponding to the effective length of each potential. It is then clear that the electronic states for all energies remain extended in the disordered system. The transmission of the whole structure is maximum for all energies since the system globally behaves as a zero potential. Let us remark that the fully resonant behavior of the Pöschl-Teller well, provided b_γ is an integer, is independent of d_γ^L, d_γ^R, and α_γ as long as the minimum value for the cutoff

distances is preserved. In fact, the real dimensional depth of the well reads $\hbar^2\alpha_\gamma^2 V_\gamma/(2m)$, hence one can choose at will the depth of the resonant well by varying α_γ, although it also means a change in the width of the potential. Therefore, one can build a disordered chain of resonant wells with different widths and depths that can even be placed at arbitrary distances from one another with absolutely no correlations in the sequence, which can be completely random indeed, and the structure will behave as a transparent potential for all energies. This is a theoretical model for which one can build totally random arrays that exhibit a full continuum of extended states and hence an interval of complete transparency: the whole positive spectrum. Let us calculate analytically the distribution of states of these disordered chains in the thermodynamic limit. For a zero potential of length L the integrated density of states is trivially $\mathcal{N}(k)=Lk/\pi$. From this fact one is led to the conclusion that a resonant well should provide the spectrum of the system with $kL_{\mathrm{eff}_\gamma}(k)/\pi$ available states with energy less than k. Since all species behave effectively as zero potentials, the IDOS of the chain per piece of length α^{-1} in the thermodynamic limit is just the composition of the contributions of the different species with their respective concentrations $\{c_\gamma\}$,

$$n(\epsilon)=\frac{1}{\pi}\sum_\gamma c_\gamma \frac{(\alpha_\gamma/\alpha)}{\alpha_\gamma(d_\gamma^R+d_\gamma^L)}\epsilon L_{\mathrm{eff}_\gamma}(\epsilon), \tag{11}$$

and the DOS would be obtained, differentiating with respect to ϵ. Inserting expression (10) into the latter definition one finally gets

$$g(\epsilon)=\frac{1}{\pi}-\frac{2}{\pi}\sum_\gamma c_\gamma \frac{(\alpha_\gamma/\alpha)}{\alpha_\gamma(d_\gamma^R+d_\gamma^L)}\sum_{j=1}^{b_\gamma-1}\frac{j(\alpha_\gamma/\alpha)}{j^2(\alpha_\gamma/\alpha)^2+\epsilon^2}. \tag{12}$$

Using the same reasoning the analytical expression for the DOS can also be straightforwardly obtained when the parameters $\{\alpha_\gamma, d_\gamma^R, d_\gamma^L\}$ obey a continuous distribution. We have carefully checked how the analytical expression reproduces exactly the distribution of states calculated numerically via the functional equation formalism. The DOS for the resonant chains is a continuous and smooth function without gaps that does not vanish for zero energy, and it registers relatively small changes by varying the concentrations or the number of different resonant wells. In Fig. 3 the tolerance of the properties of a binary resonant chain are evaluated when their parameters are deviated from the resonant values. As can be seen, a small change of the parameters mean the loss of the full resonant behavior for all energies. Nevertheless for deviations of order 1%–5% in the dimensionless amplitudes, the efficiency of transmission is still much higher than for any other nonresonant binary chain composed of wells. Naturally, for the resonant chains the Lyapunov exponent λ in the thermodynamic limit, corresponding to the inverse of the localization length, calculated via the functional equation, vanishes for all energies. It can also be checked that the inverse participation ratio for finite resonant chains as a function of the energy is simply a straight line at the value N^{-1}

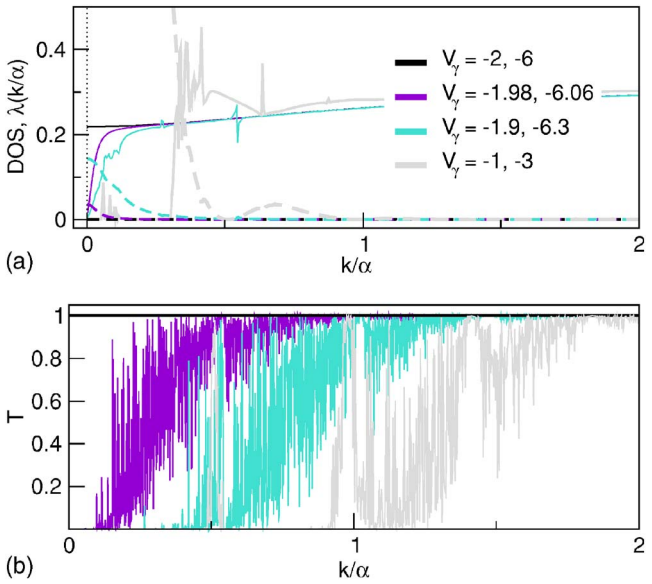

FIG. 3. (Color online) Tolerance of the properties of a binary resonant chain with their parameters. (a) DOS (solid line) and Lyapunov exponent (dashed line) in the thermodynamic limit. (b) Transmission patterns for a 1000-atom random sequence. Cases have been considered where both dimensionless amplitudes are deviated 1%, 5%, and 50% from the resonant values $V_1=-2$, $V_2=-6$. $\alpha_\gamma=\alpha$ and $d_\gamma^L=d_\gamma^R=4/\alpha$ for both species.

where N is the number of potentials, as it must be for flat extended states.

One must not forget that the transmission matrix proposed for the Pöschl-Teller potential is an approximation, since we have assumed that at the cutoff distance the asymptotic form of the states can be used. In fact, this approximation is quite correct; the error that it entails is almost irrelevant for an individual potential and the larger the cut-off distance is, the smaller the error becomes. However, it might happen that when applying the composition procedure of the potentials to

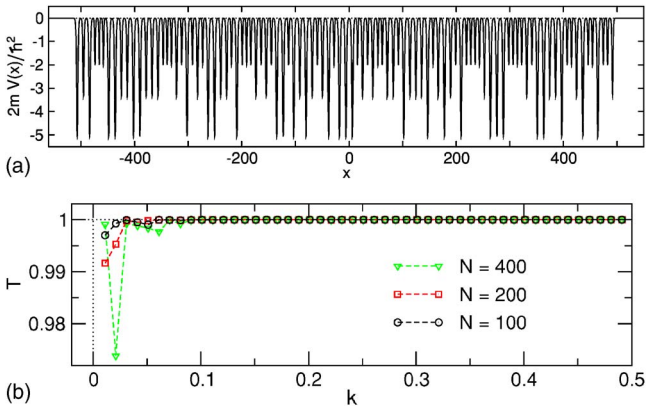

FIG. 4. (Color online) Transmission probability for random resonant chains of Pöschl-Teller wells, calculated by solving numerically the Schrödinger equation for the continuum spectrum. The upper box shows the random potential profile for 100 potentials. For all lengths the chains include three different species with symmetric cutoff ($d_\gamma^L=d_\gamma^R=d_\gamma$). The parameters are $\{\alpha_\gamma, V_\gamma, d_\gamma\}[c_\gamma]$: $\{1,-2,4\}[0.4]$, $\{0.75,-6,5.5\}[0.3]$, $\{0.65,-12,6\}[0.3]$.

build a disordered array, these small individual deviations give rise to an error growing exponentially with the length of the chain. If it were true, then the behavior of a real continuous composition of Pöschl-Teller units (i.e., the sum of all the contributions of the potentials centred at different positions) would be far from the results obtained using our techniques. In particular it would be dramatic for a resonant chain for which its resonant behavior and the delocalization of the electronic states could disappear in the real continuous composition. To show that this exponential error does not occur, we have calculated the transmission probability of several random resonant chains with 100, 200, and 400 potentials, by integrating numerically the Schrödinger equation, via a spatial discretization of the whole continuous potential of the chain, that is, taking into account the superposition of all potential wells centered at their respective positions. In Fig. 4 it can be observed how for very low energies ($k < 0.05$) a small deviation appears from $T=1$, that for the longest chain is less than 3×10^{-2}. Although this deviation seems to increase slightly with the length of the chain, its effect does not noticeably distort the resonant behavior of the chain. Then, our composition procedure describes faithfully the properties of the real continuous composite potential profile.

In summary, we have described a class of random resonant chains with a continuum of delocalized states. The composition of resonant Pöschl-Teller wells behaves as a transparent potential for all positive energies. As long as the dimensionless amplitude of the well belongs to an infinite set of discrete values that provide the resonant behavior, the rest of the parameters of the well can be varied randomly, and therefore the configuration of the resonant chain is quite versatile. And, of course, the delocalization of the electronic states for positive energies is absolutely independent of the random or correlated character of the disordered sequence. Then, at least it is possible to find a theoretical model for which disordered arrays of potentials exhibit a full continuum of extended states which is independent of the length of the system. It is in principle a pure academic model whose properties are tightly bound to the functional dependence of the potentials. Hence, its real importance depends up to a point on the possibility of reproducing experimentally such a structure. Semiconductor heterostructures may be considered as applicants for this task. Advances in the epitaxial growing techniques have made it possible to manipulate the profiles of the band conduction inside the heterostructure in order to build, for example, confining parabolic wells. If not now, perhaps in the future it might be possible to control the growing process of semiconductor samples in such a manner that the spatial profile of the band conduction follows the functional dependence of the Pöschl-Teller well and therefore has the possibility to check experimentally the predicted behavior. The Pöschl-Teller potential shows an ensemble of very interesting properties which will be described in detail[15] and also an unexpected behavior not anticipated from a disordered system.

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