On representations of solutions to certain stochastic differential equations

Javier Villarroel
Universidad de Salamanca, Facultad de Ciencias, Plaza de la Merced,37008 Salamanca, Spain
E-mail: javier@gugu.usal.es

Abstract. In this paper we obtain general conditions under which stochastic differential equations possess a strong solution representable in an explicit form as a functional of the Wiener process. Particular interest bears the problem of determining conditions that guarantee non- explosion of the solution. The necessary as well as sufficient condition is derived.

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1. Introduction

Let (Ω, G, P) be a given complete probability space, $\{W_t, t \geqslant 0\}$ a given Brownian motion defined on it with the standard filtration $F_t \equiv \sigma(W_s, s \leqslant t)$, and $x_0 \in R$ some "initial condition". Here we consider the problem of determining autonomous (i.e., time independent) stochastic differential equations (sde)

$$X_t = x_0 + \int_{t_0}^t a(X_s)ds + \int_{t_0}^t b(X_s)dW_s \tag{1}$$

that have an strong solution $X : R^+ \times \Omega \to E \subset R$ relative to (W_t, F_t) that can be expressed in the form

$$X_t(\omega) = g\left(t, \int_{t_0}^t f(t')dW_{t'}\right) \text{ a.s.} P \tag{2}$$

for some $g : R^+ \times R \to R$ and $f : R \to R$. We note that in [2] some results in this connection were obtained and the solution to Eq. (1) was reduced to solving a (deterministic) differential equation. In [4–6] Engelbert and Schmidt have considered related questions in the context of weak solutions-as opposed to strong. These formulations, however, generically do not yield explicit representations for the solutions and the semigroup generated or for other relevant properties. In [12] explicit expressions for the solution were detailed and a necessary condition that guarantees global solutions was given.

In this note we extend the above analysis corresponding to autonomous sde and we determine necessary and sufficient conditions under which Eq. (1) has a strong solution X_t of the form Eq. (2), and the main stochastic features. A major issue in our study is determining necessary and sufficient conditions that

guarantee that the solution is not exploding, and the law of the explosion time $t_{\exp}$ whenever relevant; here we define $t_{\exp}(\omega) = \sup\{t \geqslant 0 : X_t \in R\}$. and we say that explosion occurs if $P(t_{\exp} < \infty) > 0$. Under rather general conditions, only two possibilities obtain: the probability that X_t explodes in finite time is either zero or one. We determine the probability of explosion. We evaluate all the Feller's measures of the process, which allows us to classify the nature of boundaries of X_t. Recurrent properties and the existence of a stationary measure are also considered. We also determine the semigroup $(\mathbf{U}_t f)(x) \equiv E_x f(X_t)$ generated by $\{X_t\}$.

We carry out the analysis corresponding to the non autonomous or time dependent sde

$$X_t = x_0 + \int_{t_0}^t a(s, X_s)ds + \int_{t_0}^t b(s, X_s)dW_s \tag{3}$$

Assuming some differentiability on the coefficients $a, b : R^+ \times R \to R$, we give both necessary as well as sufficient conditions under which the strong solution X_t relative to (W_t, F_t) can be expressed a.s. P as a function of the form Eq. (2), and corresponding to these cases we detail that representation of the solution.

2. The time independent case; conditions for explosion

As has been already commented, in the sequel we aim to determine when does a strong solution with the representation Eq. (2) exist, and in this case, the main stochastic features of the solution. Generically, even if such a solution exists, it it might break down at a given random time, or the explosion time $t_{\exp}(\omega)$, where we define $t_{\exp}(\omega) = \inf\{t \geqslant 0 : X_t \notin R\}$. If $P(t_{\exp} < \infty) > 0$ we say that explosion occurs. An overriding issue is to determine the law of that time, and in particular give conditions that guarantee that the solution exists for all time. In this connection in [12] a condition that guarantees globality is given. The opposite issue, namely, to set forth necessary and sufficient conditions for the solution to explode in finite time and in this case, determine the distribution of the explosion time, has not been addressed at all. Here we solve these problems.

Consider the Ito Eq. (1) with W_t a Brownian motion and $b(x_0) \neq 0$. For given constants C and Q, define the difussion $\{\zeta_t, t \geqslant t_0\}$ by

$$\zeta_t \equiv \left[\tilde{W}_t + C \int_{t_0}^t f(t')dt'\right]/f; \; f(t) = e^{Qt} \tag{4}$$

where we introduce $\Delta : R \to R$ and the difussion $\tilde{W}_t$ by

$$\Delta(x) \equiv \int_{x_0}^x \frac{dx'}{b(x')}; \; \tilde{W}_t \equiv \int_{t_0}^t f(t')dW_{t'} \tag{5}$$

Define the endpoints $e_-, e_+ \in \bar{R}$ as follows: (e_-, e_+) is the biggest interval with $e_- < x_0 < e_+$ such that $\Delta(x)$ is finite $\forall x \in (e_-, e_+)$. Call exploding an endpoint e that satisfies $|\Delta(e)| < \infty$, nonexploding in the opposite case. Let τ be the first exit time of ζ_t of the interval $U \equiv (\Delta(e_-), \Delta(e_+))$. Then, the following holds.

2.1. Result 1

i] Assume that b is of class C^1 and that a, b satisfy

$$a(x) = \left[\frac{b_x}{2} + C - Q\Delta(x)\right] b(x) \tag{6}$$

$\forall x \in (e_-, e_+)$. Then, X_t defined by the implicit representation

$$\zeta_t(\omega) = \Delta(X_t(\omega)) \text{ a.s.} P, \ \omega \in \Omega \tag{7}$$

solves Eq. (1) for $t < \tau$, and $t_{\exp} = \tau$.

ii] The state space of the difussion $\{X_t, t \geqslant 0\}$ is given by $E = (e_-, e_+)$.

iii] X_t does not explode in finite time with probability 1 iff

$$|\Delta(e_-)| = \Delta(e_+) = \infty \tag{8}$$

Here we have introduced

$$\Delta(e_-) \equiv \overrightarrow{xe_-} \lim \Delta(x), \Delta(e_+) \equiv \overrightarrow{x \uparrow e_+} \lim \Delta(x) \tag{9}$$

iv] If b is of class C^2 and a is of class C^1 then a necessary condition for Eq. (1) to have an strong solution of the form Eq. (2) with g of class $C^{1,2}$ and f of class C^1 is that there exists constants C, Q such that $f(t) = e^{Qt}$ and that Eq. (6) holds locally.

2.2. Remarks

1. If b is twice differentiable and Eq. (6) holds, then a_x exists and is continuous on any closed interval contained in (e_-, e_+) and they are both locally Lipschitzian there. Strong uniqueness of the solution up to the explosion time follows.

2. Further insight in the solution can be gained by noting the following. A Wiener process $\hat{W}_t$ exists on (Ω, G, P) such that $\tilde{W}_t = \hat{W}_{\varphi(t)}$ a.s. P, where $\varphi(t) \equiv \int_{t_0}^t f^2(s)ds \equiv <\tilde{W}_t>$. Indeed $\tilde{W}_t$ is a local martingale, and the time change formula for Ito integrals- [9,13]-guarantees that $\tilde{W}_{r_t}$ is also a Brownian motion where we define $r_t \equiv \inf\{s \geqslant 0 : \varphi(s) > t\}$. (If $\varphi(\infty) < \infty$ then $\tilde{W}_{r_t}$ has the same law than Brownian motion up to time $\varphi(\infty)$). Since $\varphi(t)$ is strictly increasing it has an inverse and the result follows with $\hat{W}_t = \tilde{W}_{r_t}$.

3. When $b : R \to R^+$ is a never vanishing function of class $C^1(R)$, then $-e_- = e_+ = \infty$, Eq. (8) amounts to the condition $\int_{\pm\infty}^0 \frac{1}{b} = \infty$ and we recover the results of [12].

4. Given the standard filtration of σ-fields $G_t \equiv \sigma(X_s, 0 \leqslant s \leqslant t) \uparrow G_\infty \subset G$, we use for any G_∞ measurable $\xi : \Omega \to R$, $E_{x_0}(\xi) = E(\xi|X_0 = x_0)$. Note that $G_t \subset F_t \equiv \sigma(W_s, 0 \leqslant s \leqslant t)$ and $G_t = F_t$ if Efq. (8) holds.

2.3. Result 2

Under the conditions of result Eq. (1) and with $u_\pm \equiv \Delta(e_\pm)$, the probability that the solution does not explode in finite time is given by

i] if both endpoints are exploding

$$P_{x_0}(X_t \text{ does not explode in finite time}) = 0 \tag{10}$$

ii] if only one of the endpoints, say e_+, is exploding then *ii.1*If either $Q > 0$, or $Q = C = 0$

$$P_{x_0}\big(X_t \text{ does not explode in finite time}\big) = 0, \tag{11a}$$

ii.2] $Q = 0$.

$$P_{x_0}\big(X_t \text{ does not explode in finite time}\big) = 1 - \exp(2Cu_+), C < 0 C < 0 0, C \geqslant 0 \tag{11b}$$

ii.3] $Q < 0$. With $q \equiv \frac{C}{Q}$ one has that

$$P_{x_0}\big(X_t \text{ does not explode in finite time}\big) = \frac{\int_{-q}^{u_+ - q} e^{Qz^2} dz}{\int_{-\infty}^{u_+ - q} e^{Qz^2} dz}, \tag{11c}$$

iii] If neither point is exploding

$$P_{x_0}\big(X_t \text{ does not explode in finite time}\big) = 1 \tag{12}$$

In particular if either $Q > 0$, or $Q = C = 0$, the following zero-one law holds

$$P_{x_0}\big(X_t \text{ does not explode in finite time}\big) = 1 \Leftrightarrow |u_-| = u_+ = \infty \ 0, \text{ otherwise Eq. (13)} \tag{13}$$

Proof

By the above discussion the solution explodes at a time $t_{\exp}(\omega)$ that equals the first exit time of ζ_t of the interval U. Hence the proof is a matter of determining the corresponding probabilities for the diffusion ζ_t and will be skipped.

2.4. Recurrence properties of the process

The recurrence properties of the process Eq. (7) and a classification of its boundary points can be explicitly determined, as it is shown next. We recall several well known concepts that, after Feller [7,8], classify these matters. If B is the Borel σ- field the scale and the speed measures $S, M : B \to R$ are defined by

$$S(a, x) = \int_a^x \frac{dz}{p(z)}, M(a, x) = \int_a^x \frac{p(z)}{b^2(z)} dz; \ p(x) \equiv \exp\{2 \int^x \frac{a(z)}{b^2(z)} dz\} \tag{14a}$$

and the extension theorem. These measures are generated by distribution functions $s(x), m(x)$. Finally the Feller functions $\Sigma(a, x), \Omega(a, x)$ are defined as

$$\Sigma(a, x) = \int_a^x S(a, y) dm(y); \ \Omega(a, x) = \int_a^x M(a, y) ds(y) \tag{14b}$$

The reader is referred to [1,3,10,11] or [15].

We next evaluate the above functions and classify the boundaries of the process in terms of the signed measure $\Delta(y; x) \equiv \Delta(x) - \Delta(y)$.

2.5. Result 3

Assume the conditions of result 1 hold. If $|\Delta(e)| < \infty$ the boundary e is regular:attracting attainable. Otherwise one has according to the values of the constants C, Q that

1) If $C = Q = 0$, e is natural.
2) $Q = 0, C > 0$. e is natural if $e = e_-$, is attracting and not attainable if $e = e_+$.
3) $Q = 0, C < 0$. e is natural if $e = e_+$, is attracting and not attainable if $e = e_-$.
4) $Q > 0$. e is natural.
5) $Q < 0$. e is attracting but nonattainable .

Proof

The proof uses the fact that all Feller functions can be evaluated in an explicit way using Eq. (7). For example in the simplest case when $Q = C = 0$ is

$$M(y, x) = S(y, x) = \Delta(y, x) \equiv \Delta(x) - \Delta(y) \tag{15}$$

$$\Sigma(y, x) = \Omega(y, x) = \frac{1}{2}\Delta^2(y, x) \tag{16}$$

It is then clear that the boundary e is natural iff $\Delta(e) = \infty$. If $\Delta(e) < \infty$, using Feller's criterion we conclude that in this case there is positive probability for the boundary to be reached in finite time :$P_{x_0}\{\tau_e < \infty\} > 0$. Hence the boundary is attracting and attainable.

2.6. Ergodic properties

The situation corresponding to condition Eq. (8) bears particular interest; only under such proviso X_t is defined for all time a.s. P_{x_0}. Here we highlight some features. If either $Q = C = 0$, or $Q > 0$, then X_t is recurrent; however unlike what happens in the former, the latter is ergodic positive and a stationary distribution exists. One has.

2.7. Result 4

Assume that conditions Eq. (8) are met. Let $P_{(x;t)} : B \to R$ be the conditional probability measure on the Borel $\sigma-$field: $P_{(x;t)} \equiv P(X_t \in \bullet|X_0 = x)$, $\rho(t, y|x)$ its density. Then we have that

$$i]\rho(t, y|x) = \frac{1}{\sqrt{2\pi\Sigma^2 b^2}} \exp[-\frac{(\Delta(y) - q(1 - e^{-Qt}) - e^{-Qt}\Delta(x))^2}{2\Sigma^2}] \tag{17}$$

where

$$\Sigma^2(t) \equiv \frac{1 - e^{-2Qt}}{2Q}, q \equiv \frac{C}{Q}, Q \neq 0 \text{and} \Sigma^2 = t, q(1 - e^{-Qt}) \equiv Ct \text{if} Q = 0 \tag{18}$$

ii] If $Q > 0$, the family of probabilistic measures $\{P_{(x;t)}\}_{t \in R^+}$ is tight. Besides, as $t \to \infty$ it converges weakly to a limiting distribution $\pi : B \to R$ independent of x:

$$P_{(x;t)}\overrightarrow{t \to \infty}t \to \infty \pi, \pi(B) \equiv \int_B \sqrt{\frac{Q}{\pi b^2}} \exp[-Q(\Delta(y) - q)^2] dy \tag{19}$$

Proof
It is essentially obtained by using result 3 along with those of [14].

3. The time dependent case

Let (Ω, G, P) be a complete probability space, W_t a Brownian motion with standard filtration $F_t \equiv \sigma(W_s, s \leqslant t)$, $x_0 \in R$. Consider the Ito Eq. (3):

$$X_t(\emptyset) = x_0 + \int_{t_0}^t a(s, X_s) ds + \int_{t_0}^t b(s, X_s) dW_s$$

where $a, b : R^+ \times R \to R$ are given functions. Let

$$\Delta(t, x) \equiv \int_{x_0}^x \frac{dx'}{b(t, x')} \tag{20}$$

Define $e_{\pm}(t) \in \bar{R}$ as follows: $e_+(t) = \inf\{x : b(t, x) = 0, x > x_0\}$, $e_-(t) = \sup\{x : b(t, x) = 0, x < x_0\}$ or $e_{\pm} = \pm\infty$ if no such point exists. We aim to determine general conditions under which a strong solution X_t that can be expressed as a function of the form Eq. (2) exists. We have the following.

3.1. Result 5

i] Assume that $b(t, x)$ is of class $C^{1,1}$ for $t_0 \leqslant t < \infty, x \in R$ and that $b(t_0, x_0) \neq 0$. Let $t_1 \equiv \inf\{t > t_0 : b(t, x_0) = 0\}$. Let $Q(t)$ and $\lambda_0(t)$ be continuous functions and define

$$f(t) = \exp \int_{t_0}^t Q(s) ds, \ L(s, t) \equiv \int_s^t \lambda_0(t') dt'; L(t) \equiv L(t_0, t) \tag{21a}$$

and the difussions

$$\tilde{W}_t \equiv \int_{t_0}^t f(t') dW_{t'}; \ \zeta_t \equiv \frac{1}{f}\left[\tilde{W}_t + L(t)\right] \tag{21b}$$

Assume also that the following conditions hold for $t_0 \leqslant t < t_2$

$$\Delta(t, e_{\pm}(t)) = \pm\infty \tag{22}$$

$$a(t, x) = b(t, x)\left[\frac{b_x}{2} + \frac{\lambda_0(t)}{f(t)} - Q(t)\Delta(t, x) - \Delta_t(t, x)\right], \ x \in (e_-, e_+) \tag{23}$$

Then $X_t = \tilde{g}(t, \zeta_t) \equiv g(t, \tilde{W}_t)$ a.s. P is a strong solution to Eq. (3) relative to (W_t, F_t) for $t \in (t_0, T)$, where $T \equiv \min\{t_1, t_2\}$ and $\tilde{g}(t, z)$ is the inverse to $\Delta(t, .)$. In short

$$X_t = \Delta(t, .)^{-1}\left(\frac{1}{f(t)}\left[\tilde{W}_t + L(t)\right]\right), \text{a.s.} P \tag{24}$$

solves Eq. (3) under the condition Eq. (23).

ii] A sufficient condition for X_t not to explode in finite time with probability 1 is that $T \equiv \min\{t_1, t_2\} = \infty$, i.e., that for all time Eqs (22) and (23) hold and $b(t, x_0) \neq 0$. Otherwise blow-up at a given explosion time $t_e > T$ may occur.

iii] Assume that $b(t, x)$ and $a(t, x)$ are of class $C^{1,2}$ and $C^{0,1}$ respectively for $t_0 \leqslant t < \infty, x \in R$. Then Eq. (3) has an strong solution-up to a random time $t_{\exp}$- of the form $X_t = g(t, \int_{t_0}^t f(t')dW_{t'})$ for some g of class $C^{1,2}(\mathbf{T} \times R)$ and $f(t) \neq 0$ of class $C^1(\mathbf{T})$ iff condition Eq. (23) holds where $Q(t)$ is given by Eq. (21a) and $\lambda_0(t)$ is an arbitrary continuous functions (Here $\mathbf{T} \subset R^+$). Besides $g(t, \tilde{W}_t) \equiv \Delta(t, .)^{-1}(\zeta_t)$ is the only strong solution a.s. P.

Acknowledgments

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