ELSEVIER

1 April 2002

Physics Letters A 295 (2002) 287–298

PHYSICS LETTERS A

www.elsevier.com/locate/pla

A linear algebraic nonlinear superposition formula

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Received 15 December 2001; accepted 25 February 2002

Communicated by A.R. Bishop

Abstract

The Darboux transformation provides an iterative approach to the generation of exact solutions for an integrable system. This process can be simplified using the Bäcklund transformation and Bianchi's theorem of permutability; in this way we construct a nonlinear superposition formula, that is, an equation relating a new solution to three previous solutions. In general this equation will be a differential equation; for some examples, such as the Korteweg–de Vries equation, it is a linear algebraic equation. This last is what happens also in the case of the system discussed in this Letter. The linear algebraic nonlinear superposition formula obtained here is a new result. As an example, we use it to construct the two soliton solution, as well as special cases of this last which give rise to solutions exhibiting combinations of fission and fusion. Solutions exhibiting repeated processes of fission and fusion are new phenomena within the area of soliton equations. We also consider obtaining solutions using a symmetry approach; in this way we obtain rational solutions and also the one soliton solution.

PACS: 05.45.Yv; 02.30.Jr; 02.30.Hq

Keywords: Exact solutions; Nonlinear superposition formula

1. Introduction

One way of approaching the problem of constructing exact solutions for partial differential equations (PDEs) is the application of symmetry methods to reduce the equation to ordinary differential equations (ODEs). Another approach is to use the Darboux transformation (DT) to compute a new solution from a given one. The advantage of the last method is that it allows the possibility of iterating between solutions although it involves solving at each step for the eigenfunction of the Lax pair, i.e., solving a differential equation whose coefficients contain a previous solution of the equation.

The knowledge of the Bäcklund transformation (BT) for the system can be alternatively used to construct what is called a nonlinear superposition formula which provides a solution of the system in terms of three other known solutions. The advantage of this procedure is that this formula is in general simpler to solve than solving for the

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0375-9601/02/$ – see front matter
PII: S0375-9601(02)00189-5

eigenfunction of the Lax pair at each step. Thus, for the Korteweg–de Vries equation the nonlinear superposition formula is just an algebraic equation [1]. For other equations with higher order Lax pairs such as the ones analyzed in [2] the nonlinear superposition formula is still a linear ordinary differential equation but of lower order than the spatial part of the Lax pair. We note that nonlinear superposition formulae can also be constructed using the bilinear formalism (see, for example, [3–5]).

We consider in this Letter the problem of constructing exact solutions for the system

$$u_{xx} + w_t + \frac{1}{2}(u_x u_t)_x = 0, \tag{1}$$

$$w_x + u_{txxx} + w u_{tx} + \frac{1}{2}(u_x w_t + u_t w_x) = 0, \tag{2}$$

using both the symmetry method and the BT to construct a nonlinear superposition formula. The system (1), (2) is the first negative flow of the classical Boussinesq hierarchy and was first presented in [6] together with its Lax pair, DT and the one soliton solution. Details of the standard $(1+1)$-dimensional hierarchy in different coordinates can be found in [7–13].

The structure of the Letter is as follows. In Section 2 we apply the Lie symmetry group method to obtain exact solutions through similarity reductions. For a description of this well-known method see, for example, [14–16]. We obtain that in this case classical and nonclassical symmetries [17] provide the same similarity reductions. The travelling wave reduction yields the one soliton solution and a rational solution whereas the scaling symmetry of the system yields a fourth order ODE that passes the Painlevé test. In Section 3 we use the Lax pair and DT to obtain the BT and the nonlinear superposition formula that allows the iterative generation of solutions. We find that rather than obtaining a differential equation, this nonlinear superposition formula is, as it is for the Korteweg–de Vries equation, just a linear algebraic equation. We finally use this result to construct as an example the two soliton solution for the system which, under certain choices of the arbitrary constants, exhibits phenomena of fission and fusion. Section 4 is devoted to conclusions.

Both the nonlinear superposition formula and the two soliton solution presented here are new. Also new are our solutions of (1), (2) exhibiting fission and fusion; in particular we note that the repetition of processes of fission and fusion obtained here is a new development in soliton phenomenology.

2. Lie symmetry reductions

The application of the classical Lie group method [14–16] requires considering a one-parameter Lie group of infinitesimal transformations in the variables (x, t, u, w) given by

$$x \to x + \epsilon\xi(x,t,u,w) + O\left(\epsilon^2\right), \tag{3}$$

$$t \to t + \epsilon\tau(x,t,u,w) + O\left(\epsilon^2\right), \tag{4}$$

$$u \to u + \epsilon\phi_1(x,t,u,w) + O\left(\epsilon^2\right), \tag{5}$$

$$w \to w + \epsilon\phi_2(x,t,u,w) + O\left(\epsilon^2\right), \tag{6}$$

where ϵ is the group parameter. The condition that the above transformation leaves invariant the PDE under consideration yields an overdetermined system of linear equations for the infinitesimals $\xi(x,t,u,w)$, $\tau(x,t,u,w)$, $\phi_1(x,t,u,w)$ and $\phi_2(x,t,u,w)$. The associated Lie algebra consists then of vector fields of the form

$$\mathbf{v} = \xi(x,t,u,w)\frac{\partial}{\partial x} + \tau(x,t,u,w)\frac{\partial}{\partial t} + \phi_1(x,t,u,w)\frac{\partial}{\partial u} + \phi_2(x,t,u,w)\frac{\partial}{\partial w}. \tag{7}$$

Once the infinitesimal generators have been determined, the symmetry variables for the associated reduction can be found by solving the characteristic equations

$$\frac{dx}{\xi(x,t,u,w)} = \frac{dt}{\tau(x,t,u,w)} = \frac{du}{\phi_1(x,t,u,w)} = \frac{dw}{\phi_2(x,t,u,w)}. \tag{8}$$

For the system (1), (2) the infinitesimal are

$$\xi = c_0 x + c_1, \tag{9}$$

$$\tau = f(t), \tag{10}$$

$$\phi_1 = -2f(t) + c_2, \tag{11}$$

$$\phi_2 = -2vc_0, \tag{12}$$

where $f(t)$ is an arbitrary function of t and c_0, c_1 and c_2 are arbitrary constants. The method of Lie symmetries can be generalized to obtain what is called nonclassical or conditional symmetries (see, for example, [17]), which in principle could provide new reductions. We have evaluated the nonclassical symmetries with $\tau \neq 0$ for the system (1), (2) and we conclude that the infinitesimals are precisely the ones given above for the classical symmetries, and thus no further reductions are possible in this case.

The above infinitesimal generators provide for the system (1), (2) two different symmetry reductions, according as to whether c_0 is zero or different from zero, that we consider now in detail.

1. Case $c_0 = 0$. In this case we can set $c_1 = 1$ and $c_2 = 0$ without lost of generality. Solving the characteristic equations (8) we find the travelling wave reduction

$$u(x,t) = P(z) - 2t, \qquad w(x,t) = Q(z), \qquad z = x - F(t), \tag{13}$$

where $F(t)$ is defined as $dF(t)/dt = f(t)^{-1}$. The associated system of ODEs is

$$P_z P_{zz} + Q_z = 0, \tag{14}$$

$$P_{zzzz} + QP_{zz} + P_z Q_z = 0, \tag{15}$$

which can be written as a scalar equation after eliminating Q, and the result is

$$y_z^2 - \frac{1}{4}\left(y^4 - Ay^2 - By - C\right) = 0, \tag{16}$$

where $Q = \frac{1}{4}(A - 2y^2)$ and we have set $y(z) = P_z$ and performed three integrations (A, B and C are the three constants of integration). It is this ODE that will provide the soliton solutions. This equation can be solved in general in terms of elliptic functions. However, under certain constraints on the constants of integration we can obtain rational solutions or solutions in terms of hyperbolic functions. For $A = 6a^2$, $B = -8a^3$ and $C = 3a^4$ we obtain the solution

$$y(z) = \frac{a[a^2(z+\delta)^2 + 3]}{a^2(z+\delta)^2 - 1}, \tag{17}$$

whereas for the choice $A = 2(3\gamma^2 - 2k^2)$, $B = 8\gamma(k^2 - \gamma^2)$ and $C = \gamma^2(3\gamma^2 - 4k^2)$ the solution is

$$y(z) = \gamma - \frac{2k^2}{\gamma\cosh[k(z+\delta)] + k\sinh[k(z+\delta)] + \gamma}, \tag{18}$$

where δ is a constant of integration for both solutions. These two solutions provide solutions for the original system of PDEs by using the corresponding similarity reduction (13). In this way we obtain the rational solution

$$u = a(x - F) - 2t + 2\log\left[\frac{a(x - F + \delta) - 1}{a(x - F + \delta) + 1}\right], \tag{19}$$

$$w = \frac{a^2}{2}\left\{3 - \left[\frac{3 + a^2(x - F + \delta)^2}{1 - a^2(x - F + \delta)^2}\right]^2\right\}, \tag{20}$$

and the soliton solution

$$u = \gamma(x - F) - 2t - 2\log\left\{\frac{\gamma}{2} + \frac{k}{2}\tanh\left[\frac{k}{2}(x - F + \delta)\right]\right\}, \tag{21}$$

$$w = \frac{3}{2}\gamma^2 - k^2 - \frac{1}{2}\left\{\gamma - \frac{2k^2}{\gamma\cosh[k(x - F + \delta)] + k\sinh[k(x - F + \delta)] + \gamma}\right\}^2. \tag{22}$$

We note here that since u is in fact a potential we will be considering in what follows (u_x, w), with u and w given as above, as the one soliton solution.

2. Case $c_0 \neq 0$. We take $c_0 = 1$ and set $c_1 = c_2 = 0$ without lost of generality. The similarity variables associated to this scaling reduction are

$$u(x,t) = P(z) - 2t, \qquad w(x,t) = F(t)^2 Q(z), \qquad z = F(t)x, \tag{23}$$

where in this case $F(t)$ is defined via the expression $dF(t)/dt = -F(t)/f(t)$, and the associated system of ODEs is

$$zQ_z + zP_zP_{zz} + \frac{1}{2}P_z^2 + 2Q = 0, \tag{24}$$

$$zP_{zzzz} + zQP_{zz} + zP_zQ_z + 2QP_z + 3P_{zzz} = 0. \tag{25}$$

We can solve for Q as

$$Q = -\frac{1}{2}\frac{2z(y_{zzz} - y^2y_z) - y^3 + 6y_{zz}}{zy_z}, \tag{26}$$

where again $y(z) = P_z$, and then, after eliminating Q, we can write this system of ODEs as the following fourth order scalar ODE

$$2z^2y_zy_{zzzz} - 2z^2y_{zz}y_{zzz} + 10zy_zy_{zzz} - 6zy_{zz}^2 + 6y_zy_{zz} + zy^3y_{zz} - 6z^2yy_z^3 - 8zy^2y_z^2 - y^3y_z = 0. \tag{27}$$

It can be easily proved that the above equation passes the Painlevé test. A linear problem for the system (24), (25) is given in Appendix A. Moreover, Eq. (27) has one symmetry that we can use to integrate it once. The Lie symmetry method for ODEs as described, for example, in [16] provides the change of dependent and independent variables

$$y(z) = \xi e^{-s(\xi)}, \qquad z = e^{s(\xi)}, \tag{28}$$

which yields a third order ODE in the variable s_ξ. Setting now $s_\xi = m(\xi)/\xi$ we can reduce Eq. (27) to the third order ODE,

$$\begin{aligned}&2\xi^2m^2(m-1)m_{\xi\xi\xi} - 2\xi^2m(10m-9)m_\xi m_{\xi\xi} - 6\xi m^2(m-1)(m-2)m_{\xi\xi}\\ &\quad + 6\xi^2(5m-4)m_\xi^3 + m^2\big(4m^3 - \xi^2m^3 - 18m^2 + 26m - 12\big)m_\xi\\ &\quad + 2\xi m\big(9m^2 - 28m + 18\big)m_\xi^2 + \xi m^5(m-1)(m-2)(m-3) = 0.\end{aligned} \tag{29}$$

This equation contains no symmetries and no further integration is possible by using the symmetry approach.

3. Bäcklund transformation, nonlinear superposition formula and soliton solutions

In what follows and for reasons of convenience we will write the system (1), (2) in the form

$$u_{xx} + v_{tx} + \frac{1}{2}(u_x u_t)_x = 0, \tag{30}$$

$$v_{xx} + u_{txxx} + v_x u_{tx} + \frac{1}{2}(u_x v_{tx} + u_t v_{xx}) = 0, \tag{31}$$

where $w = v_x$. The system (30), (31) arises as the compatibility condition of the second order Lax pair [6],

$$\psi_{xx} = \frac{1}{2}(u_x - 2\lambda)\psi_x - \frac{1}{4}(v - u_x)_x\psi, \tag{32}$$

$$\psi_t = -\frac{1}{2\lambda}(u_t + 2)\psi_x - \frac{1}{4\lambda}(v - u_x)_t\psi, \tag{33}$$

where the constant λ is the spectral parameter. The DT for (30), (31) given in [6] has the form

$$\tilde{u} = u + 2\log\frac{\psi}{\psi_x}, \tag{34}$$

$$\tilde{v} = v - u_x + 2\left[\frac{\psi_x}{\psi} - \frac{1}{4}(v - u_x)_x\frac{\psi}{\psi_x}\right], \tag{35}$$

and relates two solutions (u, v) and $(\tilde{u}, \tilde{v})$ of the system (30), (31). It is well-known that the DT of an equation together with the Lax pair can be used to iterate between solutions, since given an initial solution one can solve for the eigenfunction of the Lax pair ψ and use the DT to obtain a new solution. This process requires at each step solving a differential equation. An alternative way of iterating between solutions is to apply Bianchi's theorem of permutability to the corresponding BT. We will see that in this way the process of iterating the DT is reduced for the system (30), (31) (just as it is for the Korteweg–de Vries equation) to the problem of solving a linear algebraic equation for $\tilde{u}$. As an application of this result, we will see how it can be used to obtain the two soliton solution.

The BT for the system (30), (31) can be obtained by eliminating ψ between the Lax pair (32), (33) and the DT (34); the result is,

$$p_x + p^2 - \frac{1}{2}(u_x - 2\lambda)p + \frac{1}{4}(v - u_x)_x = 0, \tag{36}$$

$$p_t + \frac{1}{2\lambda}u_{tx}p + \frac{1}{2\lambda}(u_t + 2)p_x + \frac{1}{4\lambda}(v - u_x)_{tx} = 0, \tag{37}$$

where $p = \exp[-\frac{1}{2}(\tilde{u} - u)] = \frac{\psi_x}{\psi}$. Then we have from (35) that the new solution $\tilde{v}$ can be easily written in terms of p (and so in terms of $\tilde{u}$) as

$$\tilde{v} = v - u_x + 2p - \frac{1}{2}(v - u_x)_x\frac{1}{p}, \tag{38}$$

and the problem is reduced to the one of finding an expression for $\tilde{u}$.

Let us consider the system (30), (31) and the corresponding spatial part of its BT given by (36). Let us suppose that we generate two solutions $(u_{j,1}, v_{j,1})$ and $(u_{j,2}, v_{j,2})$ of the system (30), (31) beginning with the same initial solution (u_{j-1}, v_{j-1}) but different spectral parameters λ_1 and λ_2, respectively. This then gives two different copies of Eq. (36): one with $p = \exp[-\frac{1}{2}(u_{j,1} - u_{j-1})]$ and $\lambda = \lambda_1$ and the other with $p = \exp[-\frac{1}{2}(u_{j,2} - u_{j-1})]$ and $\lambda = \lambda_2$,

$$(u_{j,1})_x = 2e^{-\frac{1}{2}(u_{j,1}-u_{j-1})} + \frac{1}{2}\big[v_{j-1} - (u_{j-1})_x\big]_x e^{\frac{1}{2}(u_{j,1}-u_{j-1})} + 2\lambda_1, \tag{39}$$

$$(u_{j,2})_x = 2e^{-\frac{1}{2}(u_{j,2}-u_{j-1})} + \frac{1}{2}\big[v_{j-1} - (u_{j-1})_x\big]_x e^{\frac{1}{2}(u_{j,2}-u_{j-1})} + 2\lambda_2. \tag{40}$$

Let us suppose now that we construct another solution $(u_{j+1,12}, v_{j+1,12})$ starting from $(u_{j,1}, v_{j,1})$ and with spectral parameter λ_2 and also a solution $(u_{j+1,21}, v_{j+1,21})$ starting from $(u_{j,2}, v_{j,2})$ and with spectral parameter λ_1. Then we can write again two copies of Eq. (36): one with $p = \exp[-\frac{1}{2}(u_{j+1,12} - u_{j,1})]$ and $\lambda = \lambda_2$ and the other one with $p = \exp[-\frac{1}{2}(u_{j+1,21} - u_{j,2})]$ and $\lambda = \lambda_1$,

$$(u_{j+1,12})_x = 2e^{-\frac{1}{2}(u_{j+1,12}-u_{j,1})} + \frac{1}{2}\big[v_{j,1} - (u_{j,1})_x\big]_x e^{\frac{1}{2}(u_{j+1,12}-u_{j,1})} + 2\lambda_2, \tag{41}$$

$$(u_{j+1,21})_x = 2e^{-\frac{1}{2}(u_{j+1,21}-u_{j,2})} + \frac{1}{2}\big[v_{j,2} - (u_{j,2})_x\big]_x e^{\frac{1}{2}(u_{j+1,21}-u_{j,2})} + 2\lambda_1. \tag{42}$$

If we now use the theorem of permutability, which states that $u_{j+1,21} = u_{j+1,12}$ (in what follows we will denote $u_{j+1,21} = u_{j+1,12} = u_{j+1}$ for simplicity), we can eliminate the first derivative of u_{j+1} between (41) and (42) and we obtain

$$G_{j+1}^2\big\{\big[v_{j,1} - (u_{j,1})_x\big]_x G_{j,1}^{-1} - \big[v_{j,2} - (u_{j,2})_x\big]_x G_{j,2}^{-1}\big\} + 4G_{j+1}(\lambda_2 - \lambda_1) + 4(G_{j,1} - G_{j,2}) = 0, \tag{43}$$

where $G_{j,1} = e^{\frac{1}{2}u_{j,1}}$, $G_{j,2} = e^{\frac{1}{2}u_{j,2}}$, and $G_{j+1} = e^{\frac{1}{2}u_{j+1}}$. The expression above is a quadratic polynomial in G_{j+1} which allows us to obtain a new solution u_{j+1} just by solving an algebraic equation.

This polynomial can still be simplified. Similarly to the four copies written before, we have in an analogous way four different copies of Eq. (38). The first two of these (those with $p = \exp[-\frac{1}{2}(u_{j,1} - u_{j-1})]$ and $p = \exp[-\frac{1}{2}(u_{j,2} - u_{j-1})]$), together with Eqs. (39) and (40) can be used to eliminate in (43) the derivatives of $(u_{j,1}, v_{j,1})$ and $(u_{j,2}, v_{j,2})$. In this way we obtain that the polynomial (43) factorizes as

$$\left[G_{j+1} - 4\frac{G_{j-1}}{(u_{j-1})_{xx} - (v_{j-1})_x}\right]\left[G_{j+1} + \frac{4G_{j-1}(G_{j,2} - G_{j,1})}{4G_{j-1}(\lambda_1 - \lambda_2) + (G_{j,2} - G_{j,1})[(u_{j-1})_{xx} - (v_{j-1})_x]}\right] = 0, \tag{44}$$

where $G_{j-1} = e^{\frac{1}{2}u_{j-1}}$. The first factor of the above expression provides a simple solution in terms of (u_{j-1}, v_{j-1}) and so in terms of just one of the three preceding solutions. It is the second factor,

$$G_{j+1} = \frac{G_{j-1}}{\frac{(\lambda_2-\lambda_1)}{(G_{j,2}-G_{j,1})}G_{j-1} - \frac{[(u_{j-1})_{xx}-(v_{j-1})_x]}{4}}, \tag{45}$$

that will allow the iterative generation of solutions, a problem which has now been reduced to the one of just solving a linear algebraic equation for G_{j+1}. In fact, using that $G_{j+1} = e^{\frac{1}{2}u_{j+1}}$ and $G_{j-1} = e^{\frac{1}{2}u_{j-1}}$, we can obtain u_{j+1} as

$$u_{j+1} = u_{j-1} - 2\log\left[\frac{(\lambda_2 - \lambda_1)}{(G_{j,2} - G_{j,1})}G_{j-1} - \frac{[(u_{j-1})_{xx} - (v_{j-1})_x]}{4}\right], \tag{46}$$

in terms of three previously known solutions (u_{j-1}, v_{j-1}), $(u_{j,1}, v_{j,1})$ and $(u_{j,2}, v_{j,2})$ of the system (30), (31). This result is new and it allows the iterative generation of solutions for this system without solving a differential equation. Moreover, since only the spatial part of the BT is used in this process, the nonlinear superposition formula (46) will hold for the entire classical Boussinesq hierarchy, and is in fact a new result for the equations of that hierarchy including for the classical Boussinesq equation itself. We note that here we concentrate on the system (30), (31) because of the new kinds of interesting classes of solutions that it possesses.

As an application of the above, we now derive the two soliton solution for the system (30), (31). In order to do so, we first need the expression for the one soliton solution that we generate now from the Lax pair and DT.

3.1. The one soliton solution

We consider in this subsection the derivation of the one soliton solution by using the Lax pair (32), (33) and the DT (34), (35). We start with the simple solution

$$u_0 = 2(\gamma+\lambda)x + g(t) - 2t, \tag{47}$$

$$v_0 = \left(\gamma^2 - k^2\right)x, \tag{48}$$

and solving for the eigenfunction of the Lax pair (32), (33) we obtain that the one soliton solution (that we will denote by (u_{1s}, v_{1s})) is given by $(\tilde{u}, \tilde{v})$ in the DT (34), (35), this is,

$$u_{1s} = u_0 + 2\log\frac{\psi}{\psi_x}, \tag{49}$$

$$v_{1s} = v_0 - (u_0)_x + 2\left[\frac{\psi_x}{\psi} - \frac{1}{4}\left[v_0 - (u_0)_x\right]_x \frac{\psi}{\psi_x}\right], \tag{50}$$

with

$$\frac{\psi}{\psi_x} = 2\frac{1+\exp\left[k\left(x - \frac{g(t)}{2\lambda} + \delta\right)\right]}{(\gamma+k)\exp\left[k\left(x - \frac{g(t)}{2\lambda} + \delta\right)\right] + (\gamma-k)}. \tag{51}$$

It is easy to see that this solution agrees with the solution (21), (22) obtained through symmetry analysis in the special case $\lambda = -\gamma/2$ and $g(t) = -\gamma F(t)$. Here however we will be taking $\lambda = -\gamma$ in the above for the one soliton solution so that u_x (the physical field in the standard classical Boussinesq system) satisfies zero boundary conditions. We note here that since both u and v are potentials we will be considering the derivatives of (u_{1s}, v_{1s}) with respect to x (and with $\lambda = -\gamma$) as the one soliton solution. We note here that the above expression for the one soliton solution when $u_0 = t$ can be found in [6].

3.2. The two soliton solution

We come back now to the problem of constructing the two soliton solution for (30), (31) by using the nonlinear superposition formula (46). We take $j = 1$ and consider the special case in which $(u_{j,1}, v_{j,1})$ and $(u_{j,2}, v_{j,2})$ are two different copies of the one soliton solution that for simplicity we will denote by (u_1, v_1) and (u_2, v_2), respectively. This means that we start with the same initial solution (u_0, v_0) and construct two copies of the one soliton solution (u_1, v_1) and (u_2, v_2) with different values of the spectral parameter (λ_1 and λ_2, respectively); since we have taken for the one soliton solution $\lambda = -\gamma$ this means different values of γ and k, say γ_i and k_i for $i = 1, 2$. These two copies will be

$$u_i = u_0(t) + 2\log\frac{\psi_i}{(\psi_i)_x}, \tag{52}$$

$$v_i = \left(\gamma_i^2 - k_i^2\right)x + 2\left[\frac{(\psi_i)_x}{\psi_i} - \frac{1}{4}\left(\gamma_i^2 - k_i^2\right)\frac{\psi_i}{(\psi_i)_x}\right], \tag{53}$$

with

$$\frac{\psi_i}{(\psi_i)_x} = 2\frac{1+\exp\left[k_i\left(x + \frac{u_0(t)}{2\gamma_i} + \frac{t}{\gamma_i} + \delta_i\right)\right]}{(\gamma_i+k_i)\exp\left[k_i\left(x + \frac{u_0(t)}{2\gamma_i} + \frac{t}{\gamma_i} + \delta_i\right)\right] + (\gamma_i - k_i)}, \tag{54}$$

for $i = 1, 2$ and where we have taken into account that $\lambda_i = -\gamma_i$, $u_0 = u_0(t)$ and $v_0 = (\gamma_1^2 - k_1^2)x = (\gamma_2^2 - k_2^2)x$ since we start with the same initial solution (u_0, v_0). The two soliton solution, that we will denote by u_{2s} is then

obtained from the nonlinear superposition formula (46)

$$u_{2s} = u_0 - 2\log\left[\frac{\gamma_1-\gamma_2}{G_2-G_1}G_0 + \frac{1}{4}(v_0)_x\right] = u_0 - 2\log\left[\frac{\gamma_1-\gamma_2}{\psi_2/(\psi_2)_x - \psi_1/(\psi_1)_x} + \frac{1}{4}(v_0)_x\right], \tag{55}$$

where we have used the definition of G_i. Finally we can write the two soliton solution as

$$u_{2s} = u_0 + 2\log\frac{\Phi}{\Phi_x}, \tag{56}$$

with

$$\frac{\Phi}{\Phi_x} = 4\frac{1 + e^{\eta_1} + e^{\eta_2} + a_{12}e^{\eta_1+\eta_2}}{b_0 + b_1e^{\eta_1} + b_2e^{\eta_2} + b_{12}e^{\eta_1+\eta_2}}, \tag{57}$$

and where $\eta_i = k_i(x + \frac{u_0}{2\gamma_i} + \frac{t}{\gamma_i} + \delta_i) + \alpha_i$. The constants α_1 and α_2 are defined by

$$e^{\alpha_1} = \left(\frac{\gamma_1+k_1-\gamma_2+k_2}{\gamma_1-k_1-\gamma_2+k_2}\right), \tag{58}$$

$$e^{\alpha_2} = \left(\frac{\gamma_2+k_2-\gamma_1+k_1}{\gamma_2-k_2-\gamma_1+k_1}\right), \tag{59}$$

and the constants in (57) are given by

$$a_{12} = \frac{(\gamma_1-k_1-\gamma_2+k_2)(\gamma_1+k_1-\gamma_2-k_2)}{(\gamma_1-k_1-\gamma_2-k_2)(\gamma_1+k_1-\gamma_2+k_2)}, \tag{60}$$

$$b_0 = (\gamma_1-k_1)(\gamma_2-k_2), \tag{61}$$

$$b_1 = (\gamma_1+k_1)(\gamma_2-k_2), \tag{62}$$

$$b_2 = (\gamma_1-k_1)(\gamma_2+k_2), \tag{63}$$

$$b_{12} = (\gamma_1+k_1)(\gamma_2+k_2)a_{12}. \tag{64}$$

The corresponding expression for v_{2s} can be now obtained from (38),

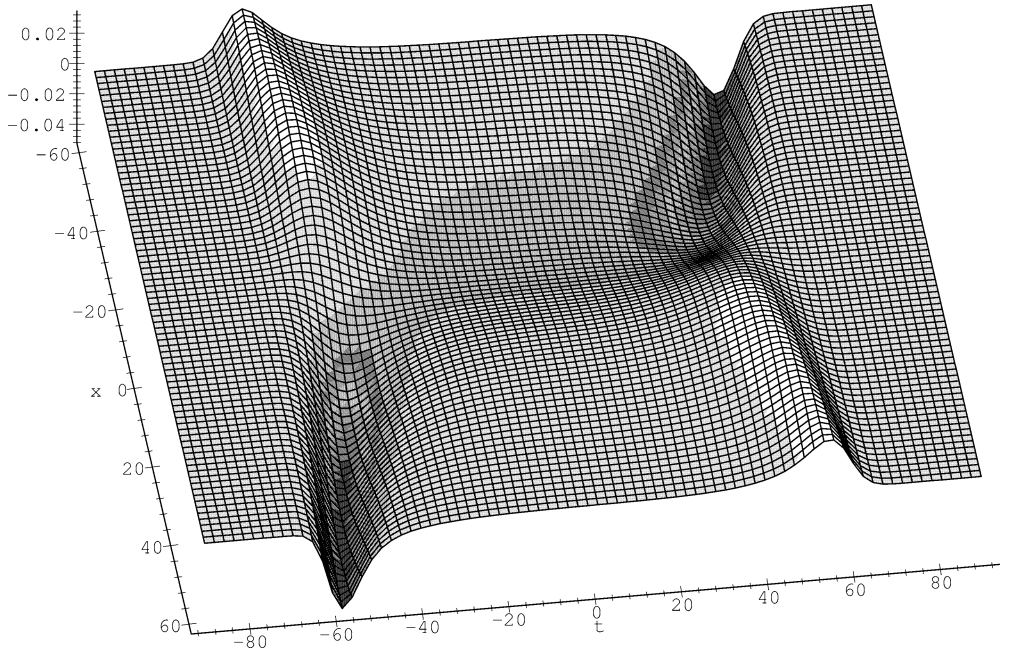
$$\begin{aligned} v_{2s} &= v_i - (u_i)_x + 2\frac{G_i}{G_0}\frac{\Phi_x}{\Phi} - \frac{1}{2}\big[v_i - (u_i)_x\big]_x\frac{G_0}{G_i}\frac{\Phi}{\Phi_x} \\ &= v_i - (u_i)_x + 2\frac{\psi_i}{(\psi_i)_x}\frac{\Phi_x}{\Phi} - \frac{1}{2}\big[v_i - (u_i)_x\big]_x\frac{(\psi_i)_x}{\psi_i}\frac{\Phi}{\Phi_x}, \end{aligned} \tag{65}$$

where we can take $i = 1$ or $i = 2$ in the above. It is important to keep in mind that the constants k_1, k_2, γ_1 and γ_2 are subject to the constraint $\gamma_1^2 - k_1^2 = \gamma_2^2 - k_2^2$ since we start with the same initial solution (u_0, v_0).

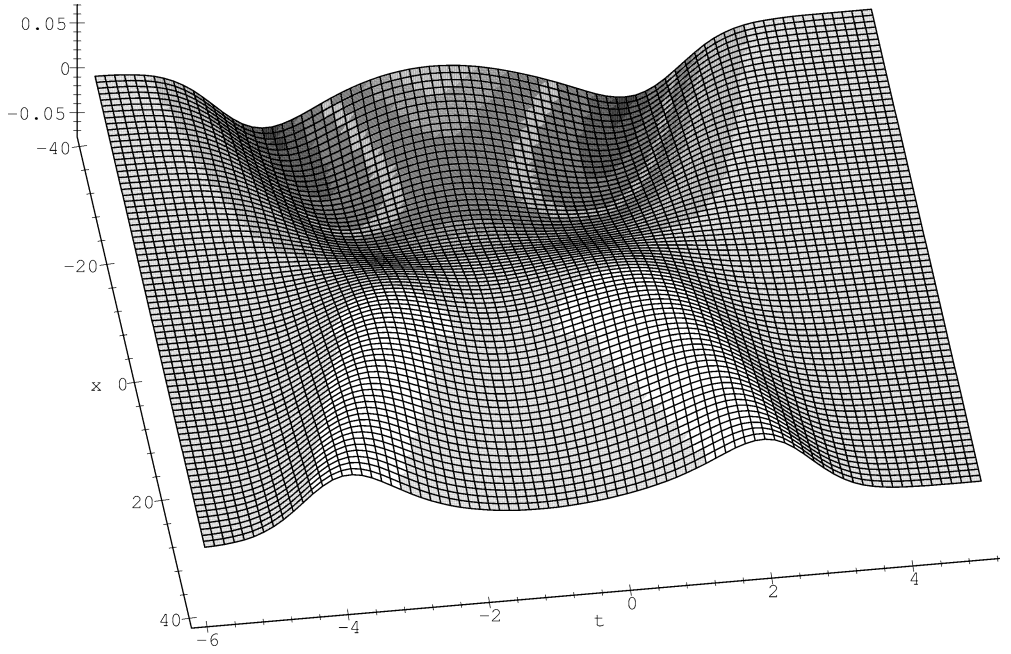
The two soliton solution given by the derivative of (56) has a rich variety of different behaviours given by the arbitrary function of time $u_0(t)$. In Fig. 1 we have plotted the two soliton solution for two specific values of u_0. Fig. 1(a) represents two solitons that interact, slow down and accelerate again after the interaction. Fig. 1(b) shows two solitons travelling in opposite directions, slowing down, interacting, and then accelerating away from the interaction with their directions reversed.

This behaviour of the two soliton solution corresponds to the general case in which the only constraint between the parameters is the one given by $\gamma_1^2 - k_1^2 = \gamma_2^2 - k_2^2$. However there are some special cases of particular interest that correspond to solutions which are also solutions of the first negative flow of the Burgers hierarchy in u_x (see [6] and [18]), i.e.,

$$u_{xx} + u_{xxt} + \frac{1}{2}(u_xu_t)_x = 0, \tag{66}$$



(a)



(b)

Fig. 1. Novel interactions of solitons. The x-derivative of solution (56) with (a) $u_0(t) = \sin(t/80)\cosh(t/15) - 2t - 1$, $\delta_1 = -\alpha_1/k_1$, $\delta_2 = -\alpha_2/k_2$, $k_1 = 0.055$, $\gamma_1 = -0.1$ and $k_2 = 0.07$. (b) $u_0(t) = \operatorname{sech}(t)\cos(t/10) + t^2$, $\delta_1 = -\alpha_1/k_1$, $\delta_2 = -\alpha_2/k_2$, $k_1 = 0.075$, $\gamma_1 = -0.1$ and $k_2 = 0.08$.

to which the system (30), (31) reduces when $v_x = u_{xx}$. They correspond to the choice $\gamma_2 = -k_2$ and $\gamma_1 = -k_1$, for which $b_{12} = a_{12} = b_1 = b_2 = 0$ and $b_0 = 4k_1k_2$. These kind of solutions represent interactions in which two incoming kinks collide to give one kink (fusion) or in which a single kink splits into two kinks (fission). We note that such behaviour is of course already known for the classical Boussinesq system itself [19–21]. It has also been explored in [22], although the results in this last appear to have different underlying explanation than those of [19–21]. Here we have an extension of the results of [19–21] whereby the system studied supports solutions in which combinations of these processes of fission and fusion are possible; thus in Fig. 2 we see that, for an appropriate choice of the function $u_0(t)$, both behaviours may occur for the same solution. Fig. 2(a) shows one kink splitting into two kinks that then fuse together into a single kink, and then split again and accelerate away from each other. Fig. 2(b) shows two incoming kinks fusing to give a single kink and then splitting again to give two kinks.

4. Conclusions

In this Letter we have considered a $(1+1)$-dimensional integrable system from the point of view of constructing exact solutions. The Lie symmetry approach provides travelling wave and scaling symmetries; we obtain with this approach rational solutions and also the one soliton solution. However, we consider of more interest the problem of iterating between solutions that we tackle by using the BT for the system in order to construct a nonlinear superposition formula. This nonlinear superposition formula consists for this system, as it does for the Korteweg–de Vries equation, of a linear algebraic equation rather than a differential equation. We thus find an explicit expression that does not involve any integration and which allows us to obtain a new solution of the system in terms of three previously known ones. Our nonlinear superposition formula holds for the entire classical Boussinesq hierarchy; as far as we are aware a linear algebraic nonlinear superposition formula is a new result for the equations of this hierarchy. As an example, we have used this formula to construct the two soliton solution and also other kinds of solutions, such as the particular case of the two soliton solution which exhibits fusion and fission of solitons. For all cases, the presence of an arbitrary function of t in the solution ensures a wide range of interesting behaviours. Solutions exhibiting repeated processes of fission and fusion constitute a new result.

Acknowledgements

The research in this Letter has been partially supported by the DGICYT under Contract No. PB98-0262. The authors would like to thank Andrew Pickering and Jose María Cerveró for helpful discussions.

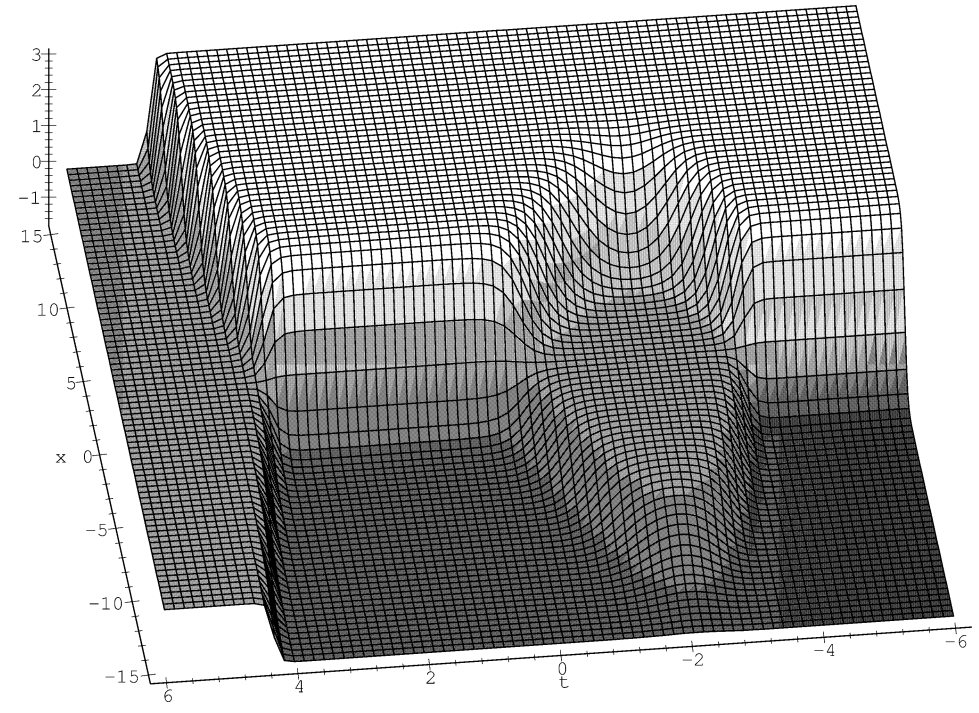
Appendix A. Linear problem for the system (24), (25)

The Lax pair for the system (32), (33) can be used to obtain a linear problem for the system of ODEs (24), (25). Using the similarity reduction (23) in the spatial part of the Lax pair (32), remembering that $w = v_x$ and setting $\mu = 2(\lambda/F)$ yields

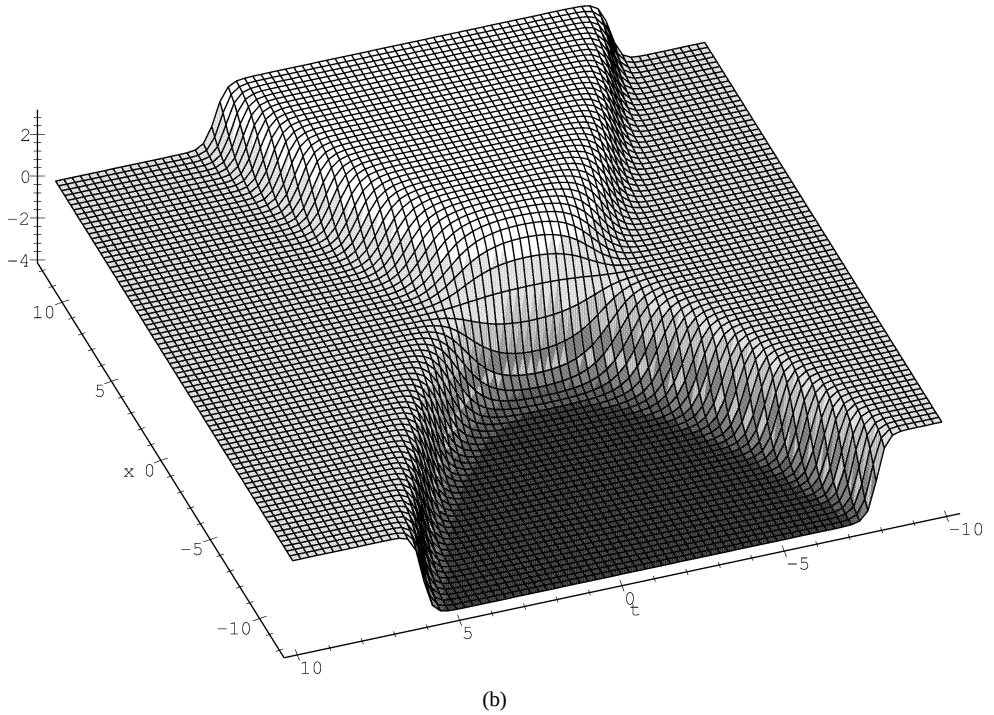
$$\psi_{zz} = \frac{1}{2}(P_z - \mu)\psi_z - \frac{1}{4}(Q - P_{zz})\psi, \tag{A.1}$$

where now $\psi = \psi(z, \mu)$. Now we use the reduction (23) in the time part of the Lax pair (33) and use the fact that in this reduction ψ is now a function of z and μ in order to compute ψ_t. The result is

$$\mu^2\psi_\mu = \big[z(\mu + P_z)\big]\psi_z + \frac{1}{2}\big[z(N - P_z)\big]_z\psi, \tag{A.2}$$



(a)



(b)

Fig. 2. Combinations of fusion and fission of solitons. The x-derivative of solution (56) with (a) $u_0(t) = t(t^2 - 15)$, $\delta_1 = -\alpha_1/k_1$, $\delta_2 = -\alpha_2/k_2$, $\gamma_1 = -k_1 = 0.85$ and $\gamma_2 = -k_2 = -1.52$. (b) $u_0(t) = t^2$, $\delta_1 = -\alpha_1/k_1$, $\delta_2 = -\alpha_2/k_2$, $\gamma_1 = -k_1 = 2$ and $\gamma_2 = -k_2 = -1.5$.

where $Q = N_z$. The compatibility condition of equations (A.1), (A.2) is precisely the system (24), (25) and so these two equations constitute a linear problem for this system. It them seems reasonable to expect that the system (24), (25)—and thus also the third order ODE (29)—will be solvable using the Inverse Monodromy Transform.

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